

On the semiduals of local isometry groups in 3d gravity

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Abstract

We use factorisations of the local isometry groups arising in 3d gravity for Lorentzian and Euclidean signatures and any value of the cosmological constant to construct associated bicrossproduct quantum groups via semidualisation. In this way we obtain quantum doubles of the Lorentz and rotation groups in 3d, as well as κ -Poincaré algebras whose associated r -matrices have spacelike, timelike and lightlike deformation parameters. We confirm and elaborate the interpretation of semiduality proposed in [13] as the exchange of the cosmological length scale and the Planck mass in the context of 3d quantum gravity. In particular, semiduality gives a simple understanding of why the quantum double of the Lorentz group and the κ -Poincaré algebra with spacelike deformation parameter are both associated to 3d gravity with vanishing cosmological constant, while the κ -Poincaré algebra with a timelike deformation parameter can only be associated to 3d gravity if one takes the Planck mass to be imaginary.

1 Introduction

1.1 Background: 3d quantum gravity and semiduality

In three dimensions (two space and one time), every solution of the Einstein equations is locally isometric to a model spacetime which is determined by the signature and the sign (or vanishing) of the cosmological constant [1]. The isometry groups of the local model spacetimes play a fundamental role in 3d gravity: elements of the isometry group provide the gluing data with which globally non-trivial solutions of the Einstein equations on a general 3-manifold are constructed from copies of the model spacetimes; in the Chern-Simons formulation of 3d gravity [2, 3], the local isometry groups play the role of gauge groups.

The space of all gluing data, suitably defined, constitutes the phase space of 3d gravity. It depends on three physical constants: the squared speed of light c^2 , the cosmological

constant Λ_C (both of which affect the structure constants of the isometry Lie algebras) and the gravitational constant G (which affects the Poisson structure). In this paper we think of the signs or vanishing of these constants as characterising different classical regimes of 3d gravity. We distinguish Lorentzian and Euclidean regimes by the sign of c^2 (we do not consider the Galilean limit here, but see [4]), regimes with positive, negative or vanishing cosmological constant, and finally regimes where the gravitational interaction is switched on or off.

Quantisation may be viewed as deformation of this classical picture into a non-commutative setting, with model spacetimes being replaced by non-commutative spaces and the local isometry groups by quantum groups. This point of view is summarised in [5], and based on the application of the combinatorial or Hamiltonian quantisation programme to the Chern-Simons formulation of 3d gravity [6, 7, 8, 9, 10, 11, 12]. The quantum group deformation of a given classical isometry group, called quantum isometry group in the following, is found via a classical r -matrix which is required to be compatible with the Cherns-Simons action in a certain sense [5].

The combinatorial quantisation procedure does not define the quantum isometry group uniquely. Instead, it defines an equivalence class of quantum groups, with equivalence essentially given by twisting. In the combinatorial approach this ambiguity does not matter since constraints are imposed after quantisation, and equivalent quantum groups give rise to the same quantum theory once the constraints are imposed. Thus, one may view combinatorial quantisation as a map from a classical regime and a given spacetime topology to a quantum theory of 3d gravity.

It is conceivable that there are equivalences between quantum groups which mean that the same quantum theory is associated to classically distinct regimes. In the interpretation of our results (though not in their derivation) we shall assume that this is not the case, i.e. we assume a one-to-one correspondence between regimes of 3d gravity and equivalence classes of quantum isometry groups. This allows us to read off the regime from a given quantum group. At the end of this introduction we shall comment on ways of establishing the validity this assumption.

An additional feature of the general picture, observed and discussed in [13] in the Euclidean setting, is that several of the quantum groups in the family of quantum doubles and bicrossproduct quantum groups associated to 3d gravity are related by a map called semidualisation. As elaborated in [13], this map, which is sometimes referred to as Born reciprocity, can be interpreted in two ways.

The first way is to think of it as an exchange of position and momentum degrees of freedom. For example, applying semiduality to the universal enveloping algebra of the Euclidean Lie algebra $\mathfrak{so}(3) \ltimes \mathbb{R}^3$, one keeps the angular momentum generators (hence the prefix semi-) and replaces momenta (which generate translations in space) by position coordinates (which generate translations in momentum space). In this simple example, the resulting algebra is isomorphic to the original one, essentially because position and momentum space are isometric in this case.

Another interpretation of semiduality, and the one that is most interesting in 3d gravity, is to interpret both the original and the semidual generators in the same way, but to think of

semiduality as a map between different regimes. An example studied in [13] is the universal enveloping algebra $U(\mathfrak{so}(4))$, which semidualises to the quantum double $D(U(\mathfrak{su}(2)))$. In 3d gravity, the former is associated with a positive cosmological constant and vanishing gravitational constant, whereas the latter is associated with vanishing cosmological constant and positive gravitational constant.

1.2 Motivation and outline

The purpose of the current paper is to explore further the interpretation and role of semiduality in 3d gravity and to extend the discussion to the Lorentzian setting. We show that the quantum doubles of $U(\mathfrak{su}(2))$ and $U(\mathfrak{sl}(2, \mathbb{R}))$ as well as the three-dimensional κ -Euclidean and κ -Poincaré algebras with timelike, lightlike and spacelike deformation parameters can all be obtained as semi-duals of the universal enveloping algebras of the local isometry Lie algebras of 3d gravity. Our calculations generalise and unify those carried out in [14] in the construction of the κ -Poincaré algebra in 3+1 dimensions [15, 16] as a bicrossproduct and those in [17, 18] in the 3d Euclidean setting.

Computing the semidual of a Hopf algebra requires that one chooses a factorisation of the original algebra as a double cross product. We consider the universal enveloping algebras of each of the isometry Lie algebras, but do not consider all possible factorisations here, leaving a systematic discussion for future work [19]. Instead, we focus on those factorisations which lead to semiduals which are either quantum doubles or κ -Euclidean and κ -Poincaré algebras. These are the quantum groups whose relationships and roles in the context of 3d gravity we would like to understand.

It turns out that the algebra structure of the semidual Hopf algebra only depends on the signature of spacetime: it is the universal enveloping algebra of the Euclidean Lie algebra in the Euclidean case, and the universal enveloping algebra of the Poincaré Lie algebra in the Lorentzian case. However, the co-algebra structure depends on the signature, the cosmological constant and also on the chosen factorisation.

We also consider the Lie bialgebras associated to each of the semidual Hopf algebras, and compute their classical r -matrices. Thus, at the end of a long chain of calculations we obtain a map from an isometry group together with a chosen factorisation to a classical r -matrix for either the Euclidean or Poincaré Lie algebra. If the factorisation does not depend on a vector, the r -matrix is always that associated to the classical double structure of the Euclidean or Poincaré Lie algebra in 3d. If the factorisation does depend on a vector, the r -matrix also depends on the vector and is of bicrossproduct type.

At the end of our paper, we summarise our results by interpreting semiduality as a map between regimes, as explained above. In the cases considered in this paper, the semidual regime always has a vanishing cosmological constant (as indicated by the algebra structure) but it may have vanishing, real or possibly imaginary gravitational constant. More precisely, it turns out that the cosmological curvature radius in the original theory (suitably interpreted in the Lorentzian case) becomes the Planck mass (which is essentially the inverse gravitational constant in 3d gravity) in the semidual theory.

Our results also suggest an interesting new viewpoint for understanding the ambiguity in determining the quantum isometry group in 3d gravity. We observe that, in the examples

considered in this paper, the r -matrices computed for semiduals of different factorisation of the same algebra are twist-equivalent. In the context of the combinatorial quantisation programme this means that those semiduals of a given isometry algebra (but for different factorisation) are equally valid as quantum isometry groups in the semidual regime. Thus we see that, in 3d gravity and for the examples considered here, twist-equivalent quantum isometry groups in the semidual regime can be understood as simply arising from different factorisations of *one* given isometry algebra in the original regime. This suggests that one may, more generally, obtain a better understanding of the twist-equivalence of quantum isometry groups by studying their semiduals.

For a full and general justification of our interpretation of semiduality as a map between physical regimes it is important to clarify if semiduality relations between quantum groups descend to semiduality relations between equivalence classes of quantum groups in the combinatorial approach to 3d quantum gravity. In order to do this one first needs to characterise (or explicitly list) the equivalence classes of quantum isometry groups in the combinatorial approach to 3d gravity. This requires, in turn, the classification of classical r -matrices which are compatible with a given Chern-Simons action. There are partial results on the classification of compatible classical r -matrices in [20] in the case of vanishing cosmological constant, and we give a full classification of the possible r -matrices compatible with a generalised Chern-Simons action for all signatures and values of the cosmological constant in a forthcoming paper [21]. However, finding all associated quantum groups presents a further and considerable challenge.

Once the equivalence classes are understood one can check if semiduality respects the equivalence. Our examples where different factorisations of one algebra give rise to twist-equivalent quantum groups suggests that this may be the case. However, many more examples or, even better, a general argument are required to clarify the picture.

This paper is organised as follows. In Sect. 2 we review the local isometry groups arising in 3d gravity and describe the associated Lie algebras in three different ways: via their Cartan decomposition, via their Iwasawa decomposition and, in a unified language for all values of the cosmological constant and both signatures, in terms of (pseudo) quaternions. Sect. 3 contains a short summary of relevant facts about bicrossproduct Hopf algebras and the mathematical process of semidualisation. In Sect. 4 we use the Iwasawa decomposition of the local isometry groups and the unified quaternionic language developed in [22] to exhibit double cross product structures of the local isometry groups. By definition, double cross product groups can be factorised into two subgroups. Factorising in different orders gives rise to actions of the two factors on each others. These transformations are key in constructing the semidual Hopf algebras and we therefore review them in Sect. 4 as well. In Sect. 5 we carry out the semidualisation for each chosen factorisation of the local isometry groups, and construct the associated bicrossproduct Hopf algebras. In our final Sect. 6 we show that the Lie bi-algebras for each of the bicrossproducts constructed via semidualisation are co-boundary, compute the associated classical r -matrices in each case and discuss our results in the general context summarised in this introduction. The appendix contains the general results of Sect. 5 in more familiar and standard notation for each signature and sign (or vanishing) of the cosmological constant.

2 Local isometry groups of 3d gravity and their Lie algebras

2.1 Conventions

The notation and concepts introduced in this section closely follow [22]. We use Einstein's summation convention and raise indices with either the three-dimensional Euclidean metric $\text{diag}(1, 1, 1)$ or the three-dimensional Minkowski metric $\text{diag}(1, -1, -1)$. A convenient way of unifying the Euclidean and Lorentzian view point is to write

$$\eta = \text{diag} \left(1, -\frac{|c|^2}{c^2}, -\frac{|c|^2}{c^2} \right). \quad (2.1)$$

Then we recover the Lorentzian metric for real speed of light c and the Euclidean metric for imaginary c . We do not introduce separate notation for the Lorentzian and the Euclidean case since the distinction will be clear from the context. In particular, we write, in either case,

$$\mathbf{p} \cdot \mathbf{q} = \eta_{ab} p^a q^b, \text{ with } \mathbf{p} = (p^0, p^1, p^2), \mathbf{q} = (q^0, q^1, q^2) \in \mathbb{R}^3. \quad (2.2)$$

The generators of both the three dimensional rotation algebra $\mathfrak{su}(2)$ and the three-dimensional Lorentz algebra $\mathfrak{sl}(2, \mathbb{R})$ are denoted by J_a , with the distinction between the two cases again given by the context. In terms of these generators the Lie brackets and Killing form are respectively

$$[J_a, J_b] = \epsilon_{abc} J^c \quad \text{and} \quad \kappa(J_a, J_b) = \eta_{ab}, \quad (2.3)$$

where ϵ denotes the fully antisymmetric tensor in three dimensions with the convention $\epsilon_{012} = \epsilon^{012} = 1$. We write \mathfrak{h} for the Lie algebra with brackets (2.3) if we do not need to distinguish between the Lorentzian and Euclidean case.

2.2 Cartan decomposition

The solutions of the Einstein equations in three dimensions are locally isometric to certain model spacetimes which are completely determined by the signature of spacetime and the cosmological constant. The isometry groups of these model spacetimes are local isometries of 3d gravity. The corresponding Lie algebras can be expressed in a common form in which the cosmological constant Λ_C and the speed of light c appear as a parameter in the Lie bracket. To achieve this, we define

$$\lambda = -c^2 \Lambda_C, \quad (2.4)$$

so that with $c = 1$ in the Lorentzian case we have $\lambda = -\Lambda_C$, and with $c = i$ in the Euclidean case we have $\lambda = \Lambda_C$. Note that the constant λ was denoted Λ in [22]. We attach the subscript C to the cosmological constant to distinguish it from the constant Λ in that paper.

We write \mathfrak{g}_λ for the family of Lie algebras arising in 3d gravity. A basis with a clear physical interpretation is the Cartan basis, consisting of generators J_a and P_a , $a = 0, 1, 2$. The J_a are the generators of the Lorentz group, where J_0 is the rotation generator, J_1 and J_2 are the boost generators and P_a are the translation generators, with Lie brackets

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c \quad \text{and} \quad [P_a, P_b] = \lambda \epsilon_{abc} J^c. \quad (2.5)$$

For $\lambda = 0$, the bracket of the generators P_a vanishes and the Lie algebra \mathfrak{g}_λ is the three-dimensional Euclidean or Poincaré Lie algebra. For $\lambda < 0$, the Lie brackets (2.5) are those of $\mathfrak{so}(3, 1) \simeq \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$. They can be obtained via the identification $P_a = i\sqrt{|\lambda|}J_a$ as the complexification of $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$ for Euclidean and Lorentzian signature, respectively. For $\lambda > 0$, the alternative generators

$$J_a^\pm = \frac{1}{2} \left(J_a \pm \frac{1}{\sqrt{\lambda}} P_a \right) \quad (2.6)$$

can be introduced in terms of which the Lie bracket takes the form of a direct sum

$$[J_a^\pm, J_b^\pm] = \epsilon_{abc} J_\pm^c, \quad [J_a^\pm, J_b^\mp] = 0. \quad (2.7)$$

Thus, the Lie algebra in this case is $\mathfrak{h} \oplus \mathfrak{h}$. For later use we also note that with $J_a = J_a^+ + J_a^-$ and $\Pi_a = \sqrt{\lambda} J_a^+$ the brackets take the form of a semidirect sum:

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, \Pi_b] = \epsilon_{abc} \Pi^c \text{ and } [\Pi_a, \Pi_b] = \sqrt{\lambda} \epsilon_{abc} \Pi^c, \quad (2.8)$$

which contracts to the Euclidean or Poincaré Lie algebra as $\lambda \rightarrow 0$.

2.3 Iwasawa decomposition

In all cases except for the Euclidean situation with $\lambda > 0$ (and hence $\Lambda_C > 0$), we can introduce a vector $\mathbf{n} = (n^0, n^1, n^2) \in \mathbb{R}^3$ satisfying

$$\mathbf{n}^2 = \eta_{ab} n^a n^b = -\lambda, \quad (2.9)$$

and define the generator \tilde{P}_a by

$$\tilde{P}_a = P_a + \epsilon_{abc} n^b J^c. \quad (2.10)$$

Then the Lie brackets on \mathfrak{g}_λ take the form

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, \tilde{P}_b] = \epsilon_{abc} \tilde{P}^c + n_b J_a - \eta_{ab} (n^c J_c), \quad [\tilde{P}_a, \tilde{P}_b] = n_a \tilde{P}_b - n_b \tilde{P}_a. \quad (2.11)$$

This shows in particular that both \mathfrak{h} and the span of $\{\tilde{P}_0, \tilde{P}_1, \tilde{P}_2\}$ form Lie subalgebras of \mathfrak{g}_λ . The latter depends on the choice of the vector \mathbf{n} . For $\mathbf{n} = 0$, which requires $\lambda = 0$, it is simply \mathbb{R}^3 with the trivial Lie bracket. For $\mathbf{n} \neq 0$, we will denote this Lie algebra by $\mathfrak{an}(2)_{\mathbf{n}}$ or, when the dependence of \mathbf{n} need not be emphasised, simply by $\mathfrak{an}(2)$. The decomposition

$$\mathfrak{g}_\lambda = \mathfrak{h} \oplus \mathfrak{an}(2)_{\mathbf{n}}, \quad (2.12)$$

implied by (2.10), generalises the Iwasawa decomposition of $\mathfrak{so}(3, 1) \simeq \mathfrak{sl}(2, \mathbb{C})$ into a compact part $\mathfrak{su}(2)$ and a non-compact part consisting of traceless, complex upper triangular matrices with real diagonal. In that context, the corresponding Lie group $AN(2)$ is the group of 2×2 matrices of the form

$$\begin{pmatrix} e^\alpha & \xi + i\eta \\ 0 & e^{-\alpha} \end{pmatrix}, \quad \alpha, \xi, \eta \in \mathbb{R}, \quad (2.13)$$

and the notation refers the abelian and the nilpotent parts of this group, which is isomorphic to the semidirect product $\mathbb{R} \ltimes \mathbb{R}^2$.

2.4 Quaternionic description

In [23], the family of Lie algebra \mathfrak{g}_λ is described in a unified fashion by identifying them with the 3d rotation and Lorentz Lie algebra over a commutative ring R_λ . The ring R_λ is a generalisation of the complex numbers, and consists of elements of the form $a + \theta b$, $a, b \in \mathbb{R}$ for a formal parameter θ satisfying $\theta^2 = \lambda$. We call a and b the real and imaginary parts, and write

$$\text{Re}_\theta(a + \theta b) = a \quad \text{Im}_\theta(a + \theta b) = b \quad \forall a, b \in \mathbb{R}. \quad (2.14)$$

The addition in R_λ is the vector space addition in \mathbb{R}^2 and multiplication rule is

$$(a + \theta b) \cdot (c + \theta d) = (ac + \lambda bd) + \theta(ad + bc) \quad \forall a, b, c, d \in \mathbb{R}. \quad (2.15)$$

There is a \mathbb{R} -linear involution $*$: $R_\lambda \mapsto R_\lambda$, defined via $(a + \theta b)^* = a - \theta b$, called θ -conjugation in the following. The ring R_λ is a field in the case $\lambda < 0$ (the complex numbers) but has zero divisors when $\lambda \geq 0$.

As shown in [22], a convenient description of the local isometry groups and their Lie algebras in 3d gravity can be obtained in terms of unit (pseudo-) quaternions over the ring R_λ . We denote the unit imaginary (pseudo-) quaternions by e_a , $a = 0, 1, 2$. They satisfy the relations

$$e_a e_b = -\eta_{ab} + \epsilon_{abc} e^c, \quad a, b = 0, 1, 2. \quad (2.16)$$

Quaternionic conjugation acts trivially on the identity 1 (often omitted when writing quaternions) and acts on imaginary quaternions according to

$$\bar{e}_a = -e_a, \quad a = 0, 1, 2. \quad (2.17)$$

It is extended linearly to a general (pseudo-) quaternions, which have the form

$$v = v_3 + v^a e_a, \quad v_0, v_1, v_2, v_3 \in \mathbb{R}. \quad (2.18)$$

If we need to distinguish between the Euclidean and the Lorentzian situation we write \mathbb{H}^E for the set of quaternions (where η_{ab} is the Euclidean metric) and \mathbb{H}^L for the set of pseudo-quaternions (where η_{ab} is the Minkowski metric), but we drop the superscript in expression which makes sense in either case. The set of unit (pseudo) quaternions is defined via

$$\mathbb{H}_1 = \{v \in \mathbb{H} | v\bar{v} = 1\}. \quad (2.19)$$

It is easy to check that $\mathbb{H}_1^E \cong SU(2)$ and $\mathbb{H}_1^L \cong SL(2, \mathbb{R})$, which motivates our notation \mathfrak{h} for the Lie algebra of either of these groups.

Combining the ring R_λ with quaternions, we define

$$\mathbb{H}^{E,L}(R_\lambda) := \mathbb{H}^{E,L} \otimes_{\mathbb{R}} R_\lambda, \quad (2.20)$$

whose elements are of the form

$$g = q_3 + \theta k_3 + (\mathbf{q} + \theta \mathbf{k}) \cdot e, \quad q_3, k_3 \in \mathbb{R}, \mathbf{q}, \mathbf{k} \in \mathbb{R}^3. \quad (2.21)$$

As shown in [22], the local isometry groups in $3d$ gravity are isomorphic to the multiplicative groups

$$\mathbb{H}_1^{E,L}(R_\lambda) := \{g \in \mathbb{H}^{E,L}(R_\lambda) | g\bar{g} = 1\} \quad (2.22)$$

of unit (pseudo) quaternions over the commutative ring R_λ . The following isomorphisms hold:

$$\begin{aligned} \mathbb{H}_1^E(R_{\lambda>0}) &\cong SU(2) \times SU(2), & \mathbb{H}_1^L(R_{\lambda>0}) &\cong SL(2, \mathbb{R}) \times SL(2, \mathbb{R}), \\ \mathbb{H}_1^E(R_{\lambda=0}) &\cong SU(2) \bowtie \mathbb{R}^3 & \mathbb{H}_1^L(R_{\lambda=0}) &\cong SL(2, \mathbb{R}) \bowtie \mathbb{R}^3, \\ \mathbb{H}_1^E(R_{\lambda<0}) &\cong SL(2, \mathbb{C}), & \mathbb{H}_1^L(R_{\lambda<0}) &\cong SL(2, \mathbb{C}). \end{aligned} \quad (2.23)$$

The Lie algebra generators P_a and J_a of \mathfrak{g}_λ can then be realised as

$$J_a = \frac{1}{2}e_a, \quad P_a = \theta J_a, \quad (2.24)$$

which reproduces the brackets (2.5). In summary, we have the following isomorphisms:

$$\begin{aligned} \mathfrak{g}_\lambda^E &\cong \begin{cases} \mathfrak{su}(2) \oplus \mathfrak{su}(2) & \text{for } \lambda > 0 \\ \mathfrak{iso}(3) & \text{for } \lambda = 0 \\ \mathfrak{so}(3, 1) & \text{for } \lambda < 0 \end{cases} \\ \mathfrak{g}_\lambda^L &\cong \begin{cases} \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) & \text{for } \lambda > 0 \\ \mathfrak{iso}(2, 1) & \text{for } \lambda = 0 \\ \mathfrak{so}(3, 1) & \text{for } \lambda < 0 \end{cases} \end{aligned} \quad (2.25)$$

2.5 Parametrising unit (pseudo-)quaternions

For the calculations in this paper involving $\lambda \neq 0$, it is useful to introduce a basis of the unit quaternions which is adapted to the sign of λ . The construction we are about to give should be thought of as a generalisation of the parametrisation of $SU(2)$ and $SL(2, \mathbb{R})$ as subsets of \mathbb{R}^4 satisfying a constraint:

$$\begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix} \in SU(2) \quad \text{iff} \quad a^2 + b^2 + c^2 + d^2 = 1, \quad (2.26)$$

and

$$\begin{pmatrix} a + b & c + d \\ -c + d & a - b \end{pmatrix} \in SL(2, \mathbb{R}) \quad \text{iff} \quad a^2 - b^2 + c^2 - d^2 = 1. \quad (2.27)$$

For the required generalisation we need to distinguish cases according to the sign of the cosmological constant Λ_C . In all calculations involving the vector \mathbf{n} and the condition (2.9) we use \mp or \pm with the upper sign referring to $\Lambda_C < 0$, i.e. $\lambda < 0$ with Euclidean signature or $\lambda > 0$ with Lorentzian signature, and the lower sign referring to $\Lambda_C > 0$, i.e. $\lambda < 0$ with Lorentzian signature (for Euclidean signature and $\Lambda_C \geq 0$ there is no non-trivial solution of (2.9)). The Lorentzian case with $\lambda = 0$ is considered separately.

For our construction we complement the vector $\mathbf{n} \in \mathbb{R}^3$ appearing in (2.9) by an orthogonal vector \mathbf{m} , satisfying $\mathbf{m}^2 = \mp \frac{1}{\lambda}$ so that $|\mathbf{m} \times \mathbf{n}|^2 = \pm 1$. Then we introduce the quaternion basis

$$\{1, \mathbf{n} \cdot \mathbf{e}, (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}, \mathbf{m} \cdot \mathbf{e}\} \quad (2.28)$$

and note that any element $v \in \mathbb{H}_1$ can be written as

$$v = a - b\mathbf{n} \cdot \mathbf{e} + c(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} + \lambda d\mathbf{m} \cdot \mathbf{e}, \quad (2.29)$$

where $a, b, c, d \in \mathbb{R}$, provided $v\bar{v} = 1$, i.e.

$$a^2 - \lambda b^2 \pm (c^2 - \lambda d^2) = 1. \quad (2.30)$$

The elements of (2.28) satisfy the following algebraic relations

$$\begin{aligned} (\mathbf{n} \cdot \mathbf{e})^2 &= \lambda, \\ ((\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e})^2 &= \mp 1, \\ (\mathbf{m} \cdot \mathbf{e})^2 &= \pm \frac{1}{\lambda}, \\ (\mathbf{n} \cdot \mathbf{e})(\mathbf{m} \cdot \mathbf{e}) &= -(\mathbf{m} \cdot \mathbf{e})(\mathbf{n} \cdot \mathbf{e}) = -(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}, \\ (\mathbf{m} \cdot \mathbf{e})((\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}) &= -((\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e})(\mathbf{m} \cdot \mathbf{e}) = \pm \frac{1}{\lambda} \mathbf{n} \cdot \mathbf{e}, \\ (\mathbf{n} \cdot \mathbf{e})((\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}) &= -((\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e})(\mathbf{n} \cdot \mathbf{e}) = -\lambda \mathbf{m} \cdot \mathbf{e}. \end{aligned} \quad (2.31)$$

It follows that the non-trivial commutators between basis elements are

$$\begin{aligned} [\mathbf{m} \cdot \mathbf{e}, \mathbf{n} \cdot \mathbf{e}] &= 2(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}, \\ [\mathbf{n} \cdot \mathbf{e}, (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}] &= -2\lambda \mathbf{m} \cdot \mathbf{e}, \\ [\lambda \mathbf{m} \cdot \mathbf{e}, (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}] &= \pm 2\mathbf{n} \cdot \mathbf{e}. \end{aligned} \quad (2.32)$$

When $\lambda = 0$ and $\mathbf{n}^2 = 0$ with $\mathbf{n} \neq 0$ (i.e in the Lorentzian case), a second lightlike vector $\tilde{\mathbf{n}}$ is introduced which satisfies

$$\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} = 0, \quad \mathbf{n} \cdot \tilde{\mathbf{n}} = 1. \quad (2.33)$$

Then the vector $\tilde{\mathbf{m}} = \tilde{\mathbf{n}} \times \mathbf{n}$ is spacelike, with $\tilde{\mathbf{m}} \cdot \tilde{\mathbf{m}} = -1$. These vectors are then used to parametrise elements $v \in \mathbb{H}_1$ via

$$v = a + b\tilde{\mathbf{m}} \cdot \mathbf{e} + \gamma \mathbf{n} \cdot \mathbf{e} + \tilde{\gamma} \tilde{\mathbf{n}} \cdot \mathbf{e} \quad (2.34)$$

in terms of $a, b, \gamma, \tilde{\gamma} \in \mathbb{R}$. The condition $v\bar{v} = 1$ is equivalent to

$$a^2 - b^2 + 2\gamma\tilde{\gamma} = 1. \quad (2.35)$$

The basis elements in this case satisfy the relations

$$\begin{aligned} (\tilde{\mathbf{m}} \cdot \mathbf{e})^2 &= 1, \\ (\tilde{\mathbf{n}} \cdot \mathbf{e})(\mathbf{n} \cdot \mathbf{e}) &= -1 + \tilde{\mathbf{m}} \cdot \mathbf{e}, \\ (\mathbf{n} \cdot \mathbf{e})(\tilde{\mathbf{n}} \cdot \mathbf{e}) &= -1 - \tilde{\mathbf{m}} \cdot \mathbf{e}, \\ (\tilde{\mathbf{n}} \cdot \mathbf{e})(\tilde{\mathbf{m}} \cdot \mathbf{e}) &= -(\tilde{\mathbf{m}} \cdot \mathbf{e})(\tilde{\mathbf{n}} \cdot \mathbf{e}) = \tilde{\mathbf{n}} \cdot \mathbf{e}, \\ (\tilde{\mathbf{m}} \cdot \mathbf{e})(\mathbf{n} \cdot \mathbf{e}) &= -(\mathbf{n} \cdot \mathbf{e})(\tilde{\mathbf{m}} \cdot \mathbf{e}) = \mathbf{n} \cdot \mathbf{e}. \end{aligned} \quad (2.36)$$

with commutation relations

$$\begin{aligned} [\tilde{\mathbf{n}} \cdot \mathbf{e}, \mathbf{n} \cdot \mathbf{e}] &= 2\tilde{\mathbf{m}} \cdot \mathbf{e}, \\ [\tilde{\mathbf{n}} \cdot \mathbf{e}, \tilde{\mathbf{m}} \cdot \mathbf{e}] &= 2\tilde{\mathbf{n}} \cdot \mathbf{e}, \\ [\tilde{\mathbf{m}} \cdot \mathbf{e}, \mathbf{n} \cdot \mathbf{e}] &= 2\mathbf{n} \cdot \mathbf{e}. \end{aligned} \quad (2.37)$$

2.6 Model spacetimes

For the calculations in this paper we will also require some background on the model spacetimes of 3d gravity. As reviewed in the introduction, these depend on the cosmological constant and the signature. A unified description of the model spacetimes in quaternionic language was given in [22]. Here we only note the conclusion that the model spacetimes can be realised as hypersurfaces in an ambient \mathbb{R}^4 , whose coordinates we denote w_0, w_1, w_2, w_3 . Then, for given signature and λ , the model spacetimes are

$$W_\lambda = \{(w_3, \mathbf{w}) \in \mathbb{R}^4 | w_3^2 + \lambda \mathbf{w}^2 = 1\}, \quad (2.38)$$

with a metric induced by the quadratic form defining the constraint

$$w_3^2 + \lambda \mathbf{w}^2 = 1. \quad (2.39)$$

As explained in [22], it is natural to think of the ambient \mathbb{R}^4 itself as the subspace of $\mathbb{H}(R_\lambda)$ which is fixed under simultaneous quaternionic and θ -conjugation. This can be made manifest by writing

$$w = w_0 + \theta \mathbf{w} \cdot \mathbf{e}, \quad (2.40)$$

which we will use later in this paper.

In the Euclidean case, W_λ is two copies of hyperbolic space when $\lambda < 0$, two copies of Euclidean space (embedded as affine spaces in \mathbb{R}^4) when $\lambda = 0$ and the 3-sphere when $\lambda > 0$. In the Lorentzian case, W_λ is a double cover of de Sitter space when $\lambda < 0$, two copies of Minkowski space (embedded as affine spaces in \mathbb{R}^4) when $\lambda = 0$ and a double cover of anti-de Sitter space when $\lambda > 0$.

2.7 Parametrisations of the $AN(2)$ group

Finally, we review two parametrisations of elements in the subgroup $AN(2)_{\mathbf{n}}$ of $\mathbb{H}_1(R_\lambda)$ which is obtained by exponentiating the Lie algebra $\mathfrak{an}(2)_{\mathbf{n}}$. Both parametrisations use quaternionic notation, and we refer to [22] for details and motivation. We omit the subscript \mathbf{n} when the dependence on \mathbf{n} is not important.

In the first parametrisation, an element $r \in AN(2)$ is given in terms of an unconstrained vector $\mathbf{q} \in \mathbb{R}^3$ by

$$r = \sqrt{1 + (\mathbf{q} \cdot \mathbf{n})^2/4} + \frac{\theta}{2} \mathbf{q} \cdot \mathbf{e} + \frac{1}{2} \mathbf{q} \times \mathbf{n} \cdot \mathbf{e}. \quad (2.41)$$

In the second, an element $r \in AN(2)$ takes the form

$$r(\alpha, z) = (1 + zQ)e^{\alpha N}, \quad (2.42)$$

where $z \in R_\lambda$ and $\alpha \in \mathbb{R}$, and N and Q are defined below. This should be thought of as a generalisation of the matrix parametrisation (2.13).

For the case $\lambda \neq 0$,

$$N = -\frac{1}{\theta} \mathbf{n} \cdot \mathbf{e}, \quad Q = \frac{\theta}{2} \mathbf{m} \cdot \mathbf{e} + \frac{1}{2} (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}. \quad (2.43)$$

Thus on setting $z = \xi + \theta\eta$, with $\xi, \eta \in \mathbb{R}$, we get

$$r(\alpha, z) = \cosh \alpha + e^{-\alpha} \left(\frac{\lambda\eta}{2} \mathbf{m} \cdot \mathbf{e} + \frac{\xi}{2} (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} \right) + \theta \left(-\frac{1}{\lambda} \mathbf{n} \cdot \mathbf{e} \sinh \alpha + e^{-\alpha} \left(\frac{\xi}{2} \mathbf{m} \cdot \mathbf{e} + \frac{\eta}{2} (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} \right) \right). \quad (2.44)$$

The parameters \mathbf{q} and (α, ξ, η) are related by

$$\mathbf{q} = -\frac{2}{\lambda} \sinh \alpha \mathbf{n} + e^{-\alpha} \xi \mathbf{m} + e^{-\alpha} \eta (\mathbf{m} \times \mathbf{n}). \quad (2.45)$$

For $\lambda = 0$, an element $r \in AN(2)$ is parametrised by $\mathbf{q} \in \mathbb{R}^3$ such that

$$\mathbf{q} = (e^{2\alpha} - 1) \tilde{\mathbf{n}} + e^{-\alpha} \xi \tilde{\mathbf{m}} + e^{-\alpha} \eta \mathbf{n} \quad (2.46)$$

and

$$r(\alpha, z) = \cosh \alpha + \mathbf{m} \cdot \mathbf{e} \sinh \alpha + e^{-\alpha} \frac{\xi}{2} \mathbf{n} \cdot \mathbf{e} + \theta \left(\tilde{\mathbf{n}} \cdot \mathbf{e} \sinh \alpha + e^{-\alpha} \left(\frac{\xi}{2} \tilde{\mathbf{m}} \cdot \mathbf{e} + \frac{\eta}{2} \mathbf{n} \cdot \mathbf{e} \right) \right). \quad (2.47)$$

Finally, we note that the coordinates α, ξ, η may also be viewed as differentiable functions

$$f : AN(2) \mapsto \mathbb{R}. \quad (2.48)$$

This is the viewpoint which we mostly adopt in this paper.

3 Bicrossproducts and semidualisation

3.1 General formalism

The calculations in this paper are based on a general method for constructing bicrossproduct Hopf algebras from factorisable Lie groups. The construction is a particular instance of the procedure of semidualisation of Hopf algebras. In the case at hand, the initial Hopf algebra is the universal enveloping algebra of one of the Lie algebras arising in 3d gravity. We briefly review those features of the construction which are required in the application to the local isometry Lie algebras of 3d gravity, and refer the reader to the original literature [24, 25] or the book [18] for a wider discussion of semidualisation. General background on Hopf algebras can be found in [18] or [26].

Suppose that X is a group (not necessarily a Lie group) which can be factorised into subgroups $G, M \subset X$ such that $X = GM$ and that the map $G \times M \rightarrow X$ given by multiplying within X is a bijection. Then every element of $x \in X$ can be uniquely expressed as a normal ordered product $x = gm$ of the elements $g \in G, m \in M$. The unique factorisation allows one to define a left action \triangleright of M on G and a right action \triangleleft of G on M

$$\triangleright : M \times G \rightarrow G, \quad \triangleleft : M \times G \rightarrow M \quad (3.1)$$

by starting with elements $g \in G, m \in M$, multiplying them in ‘wrong order’ and then factorising:

$$mg = (m \triangleright g)(m \triangleleft g), \quad \forall g \in G, m \in M. \quad (3.2)$$

These actions obey

$$e \triangleright g = g, \quad (m_1 m_2) \triangleright g = m_1 \triangleright (m_2 \triangleright g), \quad (3.3)$$

$$m \triangleright e = e, \quad m \triangleright (g_1 g_2) = (m \triangleright g_1)((m \triangleleft g_1) \triangleright g_2), \quad (3.4)$$

$$e \triangleleft g = e, \quad m \triangleleft (g_1 g_2) = (m \triangleleft g_1) \triangleleft g_2, \quad (3.5)$$

$$m \triangleleft e = m, \quad (m_1 m_2) \triangleleft g = (m_1 \triangleleft (m_2 \triangleright g))(m_2 \triangleleft g), \quad (3.6)$$

for $m_1, m_2 \in M, g_1, g_2 \in G$, with e denoting the relevant group identity element. If we want to emphasise the actions of the subgroups G and M of X on each other, we say that X is the *double cross product group* $X = G \bowtie M$.

Generally, groups G and M with actions on each other with the above properties are called a matched pair. Given a matched pair one can define the double cross product $G \bowtie M$ as the set $G \times M$ with product

$$(g_1, m_1) \cdot (g_2, m_2) = (g_1(m_1 \triangleright g_2), (m_1 \triangleleft g_2)m_2), \quad (3.7)$$

unit $e = (e, e)$ and inverse

$$(g, m)^{-1} = (m^{-1} \triangleright g^{-1}, m^{-1} \triangleleft g^{-1}) \quad (3.8)$$

and with G, M as subgroups.

As an aside we note that the above construction can be extended into the quantum group setting. Suppose then that (H_1, H_2) are a matched pair of quantum groups with $H_1 \bowtie H_2$ the associated double cross product. Then there is another quantum group denoted by $H_2 \blacktriangleleft H_1^*$, where H_1^* is the dual of H_1 , called semidualisation of the matched pair. Its dual $H_2^* \blacktriangleright H_1$ is another semidualisation, and the one we will use in this paper. Again we refer to [18] for a detailed construction.

Consider now a matched pair of groups, M and G . Assuming initially that both M and G are finite groups, we write $\mathbb{C}G$ for the group algebra of G and $\mathbb{C}(M)$ for the space of functions on M . The construction of the bicrossproduct Hopf algebra $\mathbb{C}(M) \blacktriangleright \mathbb{C}G$ then proceeds as follows. The vector space underlying the bicrossproduct $\mathbb{C}(M) \blacktriangleright \mathbb{C}G$ is the tensor product $\mathbb{C}(M) \otimes \mathbb{C}G$. Let $g \in G$ and $f \in \mathbb{C}(M)$ and consider elements of the form $f \otimes g \in \mathbb{C}(M) \otimes \mathbb{C}G$. The algebra has the multiplication \bullet , which is ordinary multiplication of the group elements and pointwise multiplication of the functions, twisted by the right action \triangleleft :

$$f_1 \otimes g_1 \bullet f_2 \otimes g_2(m) = f_1(m) f_2(m \triangleleft g_1) \otimes g_1 g_2. \quad (3.9)$$

The unit is $e \otimes 1$, where 1 is the function which is 1 everywhere on M .

In order to characterise the co-algebra structure we need to give the co-product and the co-unit. The co-product is the usual co-multiplication for the functions on the group M , but the co-product of a group element $g \in G$ is twisted by the left action \triangleright :

$$\Delta(f \otimes g)(m_1, m_2) = f(m_1 m_2) \otimes (m_2 \triangleright g) \otimes g. \quad (3.10)$$

The co-unit is $\epsilon(f \otimes g) = f(e)$. Note that we may identify $(\mathbb{C}(M) \otimes \mathbb{C}G) \otimes (\mathbb{C}(M) \otimes \mathbb{C}G)$ with $\mathbb{C}(M \times M) \otimes \mathbb{C}G \times G$ as a vector space and this is what has been done on the right hand side of (3.10) for ease of notation. The antipode is

$$S(f \otimes g)(m) = f(m^{-1} \triangleleft g^{-1}) \otimes (m^{-1} \triangleright g)^{-1}. \quad (3.11)$$

The bicrossproduct construction can be generalised to Lie group setting [24, 27, 28] under the weaker (and more useful) assumption that the original group X is factorisable into the subgroups G and M near the identity. The precise formulation requires that one defines the group algebra of a Lie group, for which there are various options, satisfying different analytical requirements. We will by-pass these issues here since we are only interested in the infinitesimal construction. More precisely, starting with a double cross sum $\mathfrak{g} \bowtie \mathfrak{m}$ of Lie algebras, we construct the semidual of the universal enveloping algebra $U(\mathfrak{g} \bowtie \mathfrak{m}) = U(\mathfrak{g}) \bowtie U(\mathfrak{m})$ as the quantum group $\mathbb{C}(M) \blacktriangleright U(\mathfrak{g})$, where $\mathbb{C}(M)$ are complex-valued smooth functions on M , dual to $U(\mathfrak{m})$. In practice, this means applying the above expressions for the Hopf algebra structures $\mathbb{C}(M) \blacktriangleright \mathbb{C}G$ to Lie groups G and M , with

$$g = e^{\varepsilon \chi}, \quad \chi \in \mathfrak{g}, \quad (3.12)$$

expanding

$$g = 1 + \varepsilon \chi + O(\varepsilon^2), \quad (3.13)$$

and keeping only leading, linear terms in ε .

Using (3.13), we obtain from (3.9)

$$\begin{aligned} [f \otimes 1, 1 \otimes \chi](m) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} ((f(m) - f(m \triangleleft e^{\varepsilon \chi}) \otimes e^{\varepsilon \chi}), \\ &= - \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(m \triangleleft e^{\varepsilon \chi}) \otimes 1, \\ [f_1 \otimes 1, f_2 \otimes 1] &= 0, \end{aligned} \quad (3.14)$$

together with the commutation relations of the Lie algebra generators $\chi \in \mathfrak{g}$. The co-products (3.10) become

$$\Delta(f \otimes \chi)(m_1, m_2) = f(m_1 m_2) \otimes \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} (m_2 \triangleright e^{\varepsilon \chi}) \otimes \chi. \quad (3.15)$$

Finally, the antipode is

$$S(f \otimes \chi)(m) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \left(f(m^{-1} \triangleleft e^{-\varepsilon \chi}) \otimes (m^{-1} \triangleright e^{\varepsilon \chi})^{-1} \right). \quad (3.16)$$

3.2 Simple examples

We consider some special cases of the general construction above, which we later apply to some of the local isometry groups in 3d gravity. Let G be a finite group and take $X = G \times G$, but viewed as a semidirect product $G \ltimes G$. This is the group analogue of the Lie algebra

decomposition (2.8) of a direct sum; see [13] for details. Then the bicrossproduct construction of the previous subsection, with $M = G$ and right action

$$h \triangleleft g = g^{-1} h g \quad (3.17)$$

as well as the trivial left action $h \triangleright g = g$ for $h, g \in G$ gives the quantum double $D(G)$. It has the algebra structure $\mathbb{C}(G) \bowtie \mathbb{C}G$ and the direct co-product

$$\Delta f \otimes g(h_1, h_2) = f(h_1 h_2) g \otimes g. \quad (3.18)$$

This is the simplest example of an interesting non-commutative and non-cocommutative Hopf algebra in the case where G is non-commutative. If G is a Lie group with Lie algebra \mathfrak{g} , then one can define the quantum double $D(U(\mathfrak{g})) = \mathbb{C}(G) \bowtie U(\mathfrak{g})$ by differentiating near the identity element in G , as illustrated above for general bicrossproducts.

Another example is the semidirect product $X = GT$ where G and T are Lie subgroups of the Lie group X and T is abelian. In this case, semidualisation leads to the Hopf algebra $T^* \bowtie U(\mathfrak{g})$, where T^* is the Pontryagin dual group (group of characters) of T , see [18].

4 Double cross product structure of local isometry groups

4.1 Factorisations of local isometry groups

Table 1 provides a list of the local isometry groups arising in 3d gravity with their corresponding matched pairs of right cross products or double cross products.

| | Euclidean signature | Lorentzian signature |
|---------------|--|---|
| $\lambda > 0$ | $\widetilde{SO}(4) = SU(2) \bowtie SU(2)$ | $\widetilde{SO}(2, 2) = \begin{cases} SL(2, \mathbb{R}) \bowtie SL(2, \mathbb{R}) \\ SL(2, \mathbb{R}) \bowtie_s AN(2) \end{cases}$ |
| $\lambda = 0$ | $\tilde{E}_3 = SU(2) \bowtie \mathbb{R}^3$ | $\tilde{P}_3 = \begin{cases} SL(2, \mathbb{R}) \bowtie \mathbb{R}^3 \\ SL(2, \mathbb{R}) \bowtie_l AN(2) \end{cases}$ |
| $\lambda < 0$ | $SL(2, \mathbb{C}) = SU(2) \bowtie AN(2)$ | $SL(2, \mathbb{C}) = SL(2, \mathbb{R}) \bowtie_t AN(2)$ |

Table 1: Local isometry groups in 3d gravity and their factorisations

Starting with Euclidean signature, the local isometry group is the ‘prototype’ double cross product $SL(2, \mathbb{C}) = SU(2) \bowtie AN(2)$, analysed in detail in [18], in the case $\lambda < 0$. When $\lambda > 0$, the local isometry group $\widetilde{SO}(4)$ can also be viewed as the semidirect product $\widetilde{SO}(4) =$

$SU(2) \bowtie SU(2)$ [13]. The corresponding Lie algebra $\mathfrak{su}(2) \bowtie \mathfrak{su}(2)$ has generators J_a and Π_a , with commutation relation (2.8). When $\lambda = 0$, the local isometry group is the double cover of the Euclidean group with the canonical factorisation given in the table.

For Lorentzian signature, a family of double cross product factorisations is implemented by (2.10) and depends on the vector \mathbf{n} , which may be spacelike, lightlike or timelike. In Table 1 we use \bowtie_s , \bowtie_l and \bowtie_t to denote the double cross products with, respectively, a spacelike, lightlike and timelike deformation vector \mathbf{n} . When $\lambda > 0$, the local isometry group $\widetilde{SO}(2, 2) = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ can also be viewed as the semidirect product $SL(2, \mathbb{R}) \bowtie SL(2, \mathbb{R})$, with Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \bowtie \mathfrak{sl}(2, \mathbb{R})$ and generators J_a and Π_a , and commutation relation (2.8). Finally, we have, as a degenerate case when $\lambda = 0$ and $\mathbf{n} = 0$, the semidirect product $SL(2, \mathbb{R}) \bowtie \mathbb{R}^3$ with Poincaré Lie algebra given by (2.5).

4.2 Right action of $U(\mathfrak{h})$ on $\mathbb{C}(AN(2))$

In the following, we compute the infinitesimal right action \triangleleft of $U(\mathfrak{h})$ on $AN(2)$ from the finite right action of \mathbb{H}_1 on $AN(2)$ given in [22], using the method outlined in Sect. 3.1. We then extract how the functions on $AN(2)$ defined in (2.48) transform under this action.

Geometrically, the right action of \mathbb{H}_1 on $AN(2)$ is the pull-back of the natural action of \mathbb{H}_1 on the model spacetimes via a map that identifies $AN(2)$ with (parts of) the model spacetime. In terms of the parametrisation $\mathbf{q} \in \mathbb{R}^3$, this identification is a map

$$S : \mathbb{R}^3 \rightarrow W_\lambda, \quad (4.1)$$

where W_λ is defined in (2.38), with the explicit form

$$S(\mathbf{q}) = (1 - \frac{\lambda}{2}\mathbf{q}^2, \sqrt{1 + \frac{(\mathbf{q} \cdot \mathbf{n})^2}{4}}\mathbf{q} + \frac{1}{2}\mathbf{q} \times (\mathbf{q} \times \mathbf{n})). \quad (4.2)$$

The conjugation action of elements v of \mathbb{H}_1 on the model spacetime is

$$I_v : W_\lambda \rightarrow W_\lambda, \quad I_v(w) = vw\bar{v}, \quad (4.3)$$

where we used the parametrisation (2.40) of elements $w \in W_\lambda$. It was shown in [22] that the inverse of S generally only exists in a subset of W_λ , but the details will not concern us here. When it exists, the formula for the inverse is

$$S^{-1}(w) = \frac{1}{\sqrt{w_3 + \mathbf{w} \cdot \mathbf{n}}} \left(\mathbf{w} + \frac{\mathbf{w}^2}{1 + w_3} \mathbf{n} \right). \quad (4.4)$$

Combining these maps, we obtain the promised right action of \mathbb{H}_1 on $AN(2)$:

$$\mathbf{q} \triangleleft v = S^{-1}(I_{v^{-1}}(S(\mathbf{q}))). \quad (4.5)$$

When $\lambda \neq 0$, substituting (2.45) into (4.2) gives

$$S(\mathbf{q}) = (w_3, \xi \mathbf{m} + \eta(\mathbf{m} \times \mathbf{n}) + \frac{1}{\lambda}(w_3 - e^{2\alpha})\mathbf{n}), \quad (4.6)$$

where

$$w_3 = \cosh 2\alpha \pm \frac{1}{2}(\xi^2 - \lambda\eta^2)e^{-2\alpha}. \quad (4.7)$$

In order to compute the commutators (3.14) it is sufficient to know the infinitesimal right action of elements $v \in \mathbb{H}_1$ near the identity. We therefore compute to linear order in the remainder of this section. We write $v = 1 + \varepsilon \mathbf{n} \cdot \mathbf{e}$ and compute:

$$I_{v^{-1}}(w) = w_3 + \theta(\mathbf{w} + 2\varepsilon(\mathbf{w} \times \mathbf{n})) \cdot \mathbf{e} + O(\varepsilon^2), \quad (4.8)$$

so that

$$I_{v^{-1}}(S(\mathbf{q})) = w_3 + \theta \left((\xi + 2\varepsilon\lambda\eta)\mathbf{m} + (\eta + 2\varepsilon\xi)(\mathbf{m} \times \mathbf{n}) + \frac{1}{\lambda}(w_3 - e^{2\alpha})\mathbf{n} \right) \cdot \mathbf{e} + O(\varepsilon^2), \quad (4.9)$$

where w_3 is defined in (4.7). The right action $r \triangleleft v$ for this case is therefore

$$S^{-1}(I_{v^{-1}}(S(\mathbf{q}))) = e^{-\alpha}(\xi + 2\varepsilon\lambda\eta)\mathbf{m} + e^{-\alpha}(\eta + 2\varepsilon\xi)(\mathbf{m} \times \mathbf{n}) - \frac{2}{\lambda} \sinh \alpha \mathbf{n} + O(\varepsilon^2). \quad (4.10)$$

This gives a new element $r \in AN(2)$ in which, to leading order in ε ,

$$\begin{aligned} \alpha &\rightarrow \alpha, \\ \xi &\rightarrow \xi + 2\varepsilon\lambda\eta, \\ \eta &\rightarrow \eta + 2\varepsilon\xi. \end{aligned} \quad (4.11)$$

Similarly, for $v = 1 + \varepsilon(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}$, we find

$$I_{v^{-1}}(w) = w_3 + \theta(\mathbf{w} + 2\varepsilon(\mathbf{w} \times (\mathbf{m} \times \mathbf{n}))) \cdot \mathbf{e} + O(\varepsilon^2), \quad (4.12)$$

which gives

$$I_{v^{-1}}(S(\mathbf{q})) = w_3 + \theta \left((\xi - 2\varepsilon(w_3 - e^{2\alpha}))\mathbf{m} + \eta(\mathbf{m} \times \mathbf{n}) + \frac{1}{\lambda}(w_3 - e^{2\alpha} \pm 2\varepsilon\xi)\mathbf{n} \right) \cdot \mathbf{e} + O(\varepsilon^2). \quad (4.13)$$

The infinitesimal right action is therefore

$$\begin{aligned} r \triangleleft v &= e^{-\alpha} (\xi + \varepsilon(e^{2\alpha} - e^{-2\alpha}) \pm \varepsilon\lambda\eta^2 e^{-2\alpha}) \mathbf{m} + e^{-\alpha} (\eta \pm \varepsilon\xi\eta e^{-2\alpha}) (\mathbf{m} \times \mathbf{n}) \\ &\quad - \frac{2}{\lambda} ((1 \pm \varepsilon\xi e^{-2\alpha}) \sinh \alpha \mp \varepsilon\xi e^{-\alpha}) \mathbf{n} + O(\varepsilon^2). \end{aligned} \quad (4.14)$$

Thus in this case, $r \triangleleft v$ transforms the parameters of r in the following manner, to linear order in ε :

$$\begin{aligned} \alpha &\rightarrow \alpha \mp \varepsilon\xi e^{-2\alpha}, \\ \xi &\rightarrow \xi + \varepsilon e^{-2\alpha} ((e^{4\alpha} - 1) \mp (\xi^2 - \lambda\eta^2)), \\ \eta &\rightarrow \eta. \end{aligned} \quad (4.15)$$

Finally, for $v = 1 + \lambda\varepsilon\mathbf{m} \cdot \mathbf{e}$, we have

$$I_{v^{-1}}(w) = w_3 + \theta(\mathbf{w} + 2\varepsilon\lambda(\mathbf{w} \times \mathbf{m})) \cdot \mathbf{e} + O(\varepsilon^2), \quad (4.16)$$

so that

$$I_{v^{-1}}(S(\mathbf{q})) = w_3 + \theta \left(\xi \mathbf{m} + (\eta - 2\varepsilon(w_3 - e^{2\alpha})(\mathbf{m} \times \mathbf{n}) + \frac{1}{\lambda}(w_3 - e^{2\alpha} \mp 2\varepsilon\lambda\eta)\mathbf{n}) \cdot \mathbf{e} + O(\varepsilon^2), \right. \\ \left. (4.17) \right.$$

and

$$r \triangleleft v = e^{-\alpha} (\xi \mp \varepsilon\lambda\xi\eta e^{-2\alpha}) \mathbf{m} + e^{-\alpha} (\eta + \varepsilon(e^{2\alpha} - e^{-2\alpha}) \mp \varepsilon\xi^2 e^{-2\alpha}) (\mathbf{m} \times \mathbf{n}) \\ - \frac{2}{\lambda} ((1 \mp \varepsilon\lambda\eta e^{-2\alpha}) \sinh \alpha \pm \varepsilon\lambda\eta e^{-\alpha}) \mathbf{n} + O(\varepsilon^2). \quad (4.18)$$

Therefore, $r \triangleleft v$ in this case has the following parameters, to linear order in ε :

$$\begin{aligned} \alpha &\rightarrow \alpha \pm \varepsilon\lambda\eta e^{-2\alpha}, \\ \xi &\rightarrow \xi, \\ \eta &\rightarrow \eta + \varepsilon e^{-2\alpha} ((e^{4\alpha} - 1) \mp (\xi^2 - \lambda\eta^2)). \end{aligned} \quad (4.19)$$

Turning to the case $\lambda = 0$, we obtain from (4.2) and (2.46)

$$S(\mathbf{q}) = \left(1, (e^{2\alpha} - 1)\tilde{\mathbf{n}} + \xi\tilde{\mathbf{m}} + e^{-2\alpha}(\eta + \frac{1}{2}\xi^2)\mathbf{n} \right). \quad (4.20)$$

Suppose $v = 1 + \varepsilon\mathbf{n} \cdot \mathbf{e}$, then

$$I_{v^{-1}}(S(\mathbf{q})) = 1 + \theta \left((e^{2\alpha} - 1)\tilde{\mathbf{n}} + (\xi + 2\varepsilon(e^{2\alpha} - 1))\tilde{\mathbf{m}} + \left(e^{-2\alpha}(\eta + \frac{1}{2}\xi^2) + 2\varepsilon\xi \right) \mathbf{n} \right) \cdot \mathbf{e}. \quad (4.21)$$

Therefore in $r \triangleleft v$, the parameters α, ξ, η transform infinitesimally according to

$$\begin{aligned} \alpha &\rightarrow \alpha, \\ \xi &\rightarrow \xi + 2\varepsilon(e^{2\alpha} - 1), \\ \eta &\rightarrow \eta + 2\varepsilon\xi. \end{aligned} \quad (4.22)$$

For $v = 1 + \varepsilon\tilde{\mathbf{m}} \cdot \mathbf{e}$,

$$I_{v^{-1}}(S(\mathbf{q})) = 1 + \theta \left((1 + 2\varepsilon)(e^{2\alpha} - 1)\tilde{\mathbf{n}} + \xi\tilde{\mathbf{m}} + e^{-2\alpha} \left((\eta + \frac{1}{2}\xi^2) - 2\varepsilon(\eta + \frac{1}{2}\xi^2) \right) \mathbf{n} \right) \cdot \mathbf{e} + O(\varepsilon^2), \quad (4.23)$$

transforming the parameters of r infinitesimally as follows:

$$\begin{aligned} \alpha &\rightarrow \alpha + \varepsilon(1 - e^{-2\alpha}), \\ \xi &\rightarrow \xi, \\ \eta &\rightarrow \eta - 2\varepsilon(\eta + \frac{1}{2}\xi^2)e^{-2\alpha}. \end{aligned} \quad (4.24)$$

Finally, for $v = 1 + \varepsilon\tilde{\mathbf{n}} \cdot \mathbf{e}$ we have

$$I_{v^{-1}}(S(\mathbf{q})) = 1 + \theta \left((e^{2\alpha} - 1 - 2\varepsilon\xi)\tilde{\mathbf{n}} + \left(\xi - 2\varepsilon e^{-2\alpha}(\eta + \frac{1}{2}\xi^2) \right) \tilde{\mathbf{m}} + e^{-2\alpha}(\eta + \frac{1}{2}\xi^2)\mathbf{n} \right) \cdot \mathbf{e} + O(\varepsilon^2) \quad (4.25)$$

and the parameters of $r \in AN(2)$ under the right action transform infinitesimally according to

$$\begin{aligned} \alpha &\rightarrow \alpha - \varepsilon\xi e^{-2\alpha}, \\ \xi &\rightarrow \xi - 2\varepsilon e^{-2\alpha}(\eta + \frac{1}{2}\xi^2), \\ \eta &\rightarrow \eta. \end{aligned} \quad (4.26)$$

4.3 Left action of $AN(2)$ on $U(\mathfrak{h})$

If $\lambda \neq 0$, we define the projections

$$P = \frac{1}{2}(1 + N) \text{ and } \bar{P} = \frac{1}{2}(1 - N), \quad (4.27)$$

where $N = -\frac{1}{\theta}\mathbf{n} \cdot \mathbf{e}$. Then the left action of an element $r \in AN(2)$ on an element $v \in U(\mathfrak{h})$ is given in [22] as

$$r \triangleright v = \frac{1}{N_-}(rvP + r^*v\bar{P}) \quad (4.28)$$

where the normalisation factor

$$N_- = |rvP + r^*v\bar{P}| \quad (4.29)$$

ensures that $r \triangleright v$ is a unit (pseudo) quaternion. We have written $|q| = \sqrt{q\bar{q}}$ for the ‘norm’ of a quaternion here. In the Lorentzian case this norm could be ill-defined or zero, but in our applications this will not concern us since v will be near the identity (and hence has well-defined ‘norm’) and we only consider infinitesimal changes.

Again we derive the infinitesimal version. Suppose $v = 1 + \varepsilon\mathbf{n} \cdot \mathbf{e}$, with ε infinitesimal. Then putting (2.44) and (4.27) into (4.28) and using the properties in (2.31), we see that the left action of r on v leaves v invariant, i.e.

$$r \triangleright v = 1 + \varepsilon\mathbf{n} \cdot \mathbf{e}. \quad (4.30)$$

Next, we take $v = 1 + \varepsilon(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}$. The left action (4.28) then simplifies to

$$r \triangleright v = 1 + \varepsilon(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} e^{-2\alpha} \pm \varepsilon\mathbf{n} \cdot \mathbf{e} \eta e^{-2\alpha} + O(\varepsilon^2) \quad (4.31)$$

on using properties (2.31). Considering $v = 1 + \lambda\varepsilon\mathbf{m} \cdot \mathbf{e}$ the left action of r on v can easily be obtained in a similar fashion. Here,

$$r \triangleright v = 1 + \varepsilon\lambda\mathbf{m} \cdot \mathbf{e} e^{-2\alpha} \pm \varepsilon\mathbf{n} \cdot \mathbf{e} \xi e^{-2\alpha} + O(\varepsilon^2). \quad (4.32)$$

Next we consider $\lambda = 0$ and $\mathbf{n} \neq 0$. The left action $r \triangleright v$ of $r \in AN(2)$ on $v \in U(\mathfrak{h})$ is given in [22] as

$$\begin{aligned} r \triangleright v &= \frac{1}{2N_-} \left((rv + r^*v) - (rv - r^*v) \frac{\mathbf{n} \cdot \mathbf{e}}{\theta} \right) \\ &= \frac{1}{N_-} (\text{Re}_\theta(r)v - \text{Im}_\theta(r)v\mathbf{n} \cdot \mathbf{e}) \end{aligned} \quad (4.33)$$

with normalisation factor

$$N_- = |\text{Re}_\theta(r)v - \text{Im}_\theta(r)v\mathbf{n} \cdot \mathbf{e}|, \quad (4.34)$$

where the comments made after (4.29) apply again. When $v = 1 + \varepsilon\mathbf{n} \cdot \mathbf{e}$, we have

$$r \triangleright v = 1 + \varepsilon\mathbf{n} \cdot \mathbf{e}, \quad (4.35)$$

where we have used properties (2.36). For $v = 1 + \varepsilon\tilde{\mathbf{m}} \cdot \mathbf{e}$, we obtain from (4.33)

$$r \triangleright v = 1 + \varepsilon\tilde{\mathbf{m}} \cdot \mathbf{e} e^{-2\alpha} - \varepsilon\mathbf{n} \cdot \mathbf{e} \xi e^{-2\alpha} + O(\varepsilon^2). \quad (4.36)$$

Finally, for $v = 1 + \varepsilon\tilde{\mathbf{n}} \cdot \mathbf{e}$ the leading terms in the left action are

$$r \triangleright v = 1 + \varepsilon\tilde{\mathbf{n}} \cdot \mathbf{e} e^{-2\alpha} + \varepsilon\mathbf{n} \cdot \mathbf{e} \eta e^{-2\alpha} + O(\varepsilon^2). \quad (4.37)$$

5 Bicrossproduct quantum groups in 3d gravity

We now combine the general results (3.14)-(3.16) of Sect. 3.1 with the left and right actions computed in the previous sections to obtain the Hopf algebraic structures of the bicrossproduct quantum group $\mathbb{C}(AN) \bowtie U(\mathfrak{h})$.

5.1 Algebra structure

Writing simply α for $\alpha \otimes 1$ and similarly $\mathbf{n} \cdot \mathbf{e}$ for $1 \otimes \mathbf{n} \cdot \mathbf{e}$ etc., we obtain, from (3.14), the following commutation relations in the case $\lambda \neq 0$:

$$\begin{aligned}
[\alpha, \xi] &= [\alpha, \eta] = [\alpha, \mathbf{n} \cdot \mathbf{e}] = 0, \\
[\eta, (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}] &= [\xi, \lambda \mathbf{m} \cdot \mathbf{e}] = 0, \\
[\xi, \mathbf{n} \cdot \mathbf{e}] &= -2\lambda\eta, \\
[\eta, \mathbf{n} \cdot \mathbf{e}] &= -2\xi, \\
[\alpha, (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}] &= \pm \xi e^{-2\alpha}, \\
[\alpha, \lambda \mathbf{m} \cdot \mathbf{e}] &= \mp \lambda \eta e^{-2\alpha}, \\
[\xi, (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}] &= -e^{-2\alpha} ((e^{4\alpha} - 1) \mp (\xi^2 - \lambda\eta^2)), \\
[\eta, \lambda \mathbf{m} \cdot \mathbf{e}] &= -e^{-2\alpha} ((e^{4\alpha} - 1) \mp (\xi^2 - \lambda\eta^2)),
\end{aligned} \tag{5.1}$$

together with the commutation relations (2.32).

When $\lambda = 0$, the algebra has the commutation relation (2.37) and

$$\begin{aligned}
[\alpha, \xi] &= [\alpha, \eta] = [\alpha, \mathbf{n} \cdot \mathbf{e}] = [\xi, \tilde{\mathbf{m}} \cdot \mathbf{e}] = [\eta, \tilde{\mathbf{n}} \cdot \mathbf{e}] = 0, \\
[\xi, \mathbf{n} \cdot \mathbf{e}] &= -2(e^{2\alpha} - 1), \quad [\alpha, \tilde{\mathbf{m}} \cdot \mathbf{e}] = e^{-2\alpha} - 1, \\
[\eta, \mathbf{n} \cdot \mathbf{e}] &= -2\xi, \quad [\alpha, \tilde{\mathbf{n}} \cdot \mathbf{e}] = \xi e^{-2\alpha}, \\
[\xi, \tilde{\mathbf{n}} \cdot \mathbf{e}] &= 2e^{-2\alpha}(\eta + \tfrac{1}{2}\xi^2), \quad [\eta, \tilde{\mathbf{m}} \cdot \mathbf{e}] = 2(\eta + \tfrac{1}{2}\xi^2)e^{-2\alpha}.
\end{aligned} \tag{5.2}$$

5.2 Co-algebra Structure and Antipodes

Next, we calculate the co-products and antipodes. Suppose

$$r_1(\alpha_1, z_1) = (1 + z_1 Q) e^{\alpha_1 N} \text{ and } r_2(\alpha_2, z_2) = (1 + z_2 Q) e^{\alpha_2 N}, \tag{5.3}$$

then

$$r_1 r_2 = (1 + (z_1 + e^{2\alpha_1} z_2) Q) e^{(\alpha_1 + \alpha_2) N}. \tag{5.4}$$

Hence from (2.48), if we choose the continuous function $f \in \mathbb{C}(AN(2))$ to be $f(r) = \alpha$, then $f(r_1 r_2) = \alpha_1 + \alpha_2$, and (3.15) gives

$$\Delta \alpha = \alpha \otimes 1 + 1 \otimes \alpha. \tag{5.5}$$

Also, with the choice of $f(r) = z$, we have

$$f(r_1 r_2) = z_1 + e^{2\alpha_1} z_2 \tag{5.6}$$

and (3.15) gives

$$\Delta z = z \otimes 1 + e^{2\alpha} \otimes z. \tag{5.7}$$

Using the definition in (3.16) we get the antipode

$$S(\alpha) = -\alpha \quad (5.8)$$

$$S(z) = -e^{-2\alpha}z. \quad (5.9)$$

When $\lambda \neq 0$, we again consider $v = 1 + \varepsilon \mathbf{n} \cdot \mathbf{e}$. Then (4.28) and (3.15) give the co-product

$$\Delta v = 1 \otimes 1 + \varepsilon(1 \otimes \mathbf{n} \cdot \mathbf{e} + \mathbf{n} \cdot \mathbf{e} \otimes 1) + O(\varepsilon^2), \quad (5.10)$$

and the antipode is given by (3.16) as

$$S(v) = 1 - \varepsilon \mathbf{n} \cdot \mathbf{e}. \quad (5.11)$$

When $v = 1 + \varepsilon(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}$, the co-product of v is given by (3.15) as

$$\Delta v = 1 \otimes 1 + \varepsilon(1 \otimes (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} + (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} \otimes e^{-2\alpha} \pm \mathbf{n} \cdot \mathbf{e} \otimes \eta e^{-2\alpha}) + O(\varepsilon^2). \quad (5.12)$$

The antipode is

$$S(v) = 1 - \varepsilon(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} e^{2\alpha} \pm \varepsilon \mathbf{n} \cdot \mathbf{e} \eta + O(\varepsilon^2). \quad (5.13)$$

For $v = 1 + \lambda \varepsilon \mathbf{m} \cdot \mathbf{e}$, the co-product is

$$\Delta v = 1 \otimes 1 + \varepsilon(1 \otimes \lambda \mathbf{m} \cdot \mathbf{e} + \lambda \mathbf{m} \cdot \mathbf{e} \otimes e^{-2\alpha} \pm \mathbf{n} \cdot \mathbf{e} \otimes \xi e^{-2\alpha}) + O(\varepsilon^2), \quad (5.14)$$

and the antipode is

$$S(v) = 1 - \varepsilon \lambda \mathbf{m} \cdot \mathbf{e} e^{2\alpha} \pm \varepsilon \mathbf{n} \cdot \mathbf{e} \xi + O(\varepsilon^2). \quad (5.15)$$

For $\lambda = 0$ and $\mathbf{n} \neq 0$, we take $v = 1 + \varepsilon \mathbf{n} \cdot \mathbf{e}$ and find the co-product

$$\Delta v = 1 \otimes 1 + \varepsilon(1 \otimes \mathbf{n} \cdot \mathbf{e} + \mathbf{n} \cdot \mathbf{e} \otimes 1) + O(\varepsilon^2), \quad (5.16)$$

and the antipode is given from (3.16) by

$$S(v) = 1 - \varepsilon \mathbf{n} \cdot \mathbf{e}. \quad (5.17)$$

When $v = 1 + \varepsilon \tilde{\mathbf{m}} \cdot \mathbf{e}$, the co-product is

$$\Delta v = 1 \otimes 1 + \varepsilon(1 \otimes \tilde{\mathbf{m}} \cdot \mathbf{e} + \tilde{\mathbf{m}} \cdot \mathbf{e} \otimes e^{-2\alpha} - \varepsilon \mathbf{n} \cdot \mathbf{e} \otimes \xi e^{-2\alpha}) + O(\varepsilon^2), \quad (5.18)$$

and the antipode is

$$S(v) = 1 - \varepsilon \tilde{\mathbf{m}} \cdot \mathbf{e} e^{2\alpha} + \varepsilon \mathbf{n} \cdot \mathbf{e} \xi + O(\varepsilon^2). \quad (5.19)$$

Finally, for $v = 1 + \varepsilon \tilde{\mathbf{n}} \cdot \mathbf{e}$, the co-product is

$$\Delta v = 1 \otimes 1 + \varepsilon(1 \otimes \tilde{\mathbf{n}} \cdot \mathbf{e} + \tilde{\mathbf{n}} \cdot \mathbf{e} \otimes e^{-2\alpha} + \mathbf{n} \cdot \mathbf{e} \otimes \eta e^{-2\alpha}) + O(\varepsilon^2), \quad (5.20)$$

and the antipode is

$$S(v) = 1 - \varepsilon \tilde{\mathbf{n}} \cdot \mathbf{e} e^{2\alpha} + \varepsilon \mathbf{n} \cdot \mathbf{e} \eta + O(\varepsilon^2). \quad (5.21)$$

We now summarise the above results. For $\lambda \neq 0$, the co-products are given by

$$\begin{aligned}
\Delta\alpha &= \alpha \otimes 1 + 1 \otimes \alpha, \\
\Delta z &= z \otimes 1 + e^{2\alpha} \otimes z, \\
\Delta\mathbf{n} \cdot \mathbf{e} &= 1 \otimes \mathbf{n} \cdot \mathbf{e} + \mathbf{n} \cdot \mathbf{e} \otimes 1, \\
\Delta(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} &= 1 \otimes (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} + (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} \otimes e^{-2\alpha} \pm \mathbf{n} \cdot \mathbf{e} \otimes \eta e^{-2\alpha}, \\
\Delta\lambda\mathbf{m} \cdot \mathbf{e} &= 1 \otimes \lambda\mathbf{m} \cdot \mathbf{e} + \lambda\mathbf{m} \cdot \mathbf{e} \otimes e^{-2\alpha} \mp \mathbf{n} \cdot \mathbf{e} \otimes \xi e^{-2\alpha},
\end{aligned} \tag{5.22}$$

and the antipode is

$$\begin{aligned}
S(\alpha) &= -\alpha, \\
S(z) &= -e^{-2\alpha} z, \\
S(\mathbf{n} \cdot \mathbf{e}) &= -\mathbf{n} \cdot \mathbf{e}, \\
S((\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}) &= -(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} e^{2\alpha} \pm \mathbf{n} \cdot \mathbf{e} \eta, \\
S(\lambda\mathbf{m} \cdot \mathbf{e}) &= -\lambda\mathbf{m} \cdot \mathbf{e} e^{2\alpha} \pm \mathbf{n} \cdot \mathbf{e} \xi.
\end{aligned} \tag{5.23}$$

In the case $\lambda = 0$, co-products and antipodes are given respectively by

$$\begin{aligned}
\Delta\alpha &= \alpha \otimes 1 + 1 \otimes \alpha, \\
\Delta z &= z \otimes 1 + e^{2\alpha} \otimes z, \\
\Delta\mathbf{n} \cdot \mathbf{e} &= 1 \otimes \mathbf{n} \cdot \mathbf{e} + \mathbf{n} \cdot \mathbf{e} \otimes 1, \\
\Delta\tilde{\mathbf{m}} \cdot \mathbf{e} &= 1 \otimes \tilde{\mathbf{m}} \cdot \mathbf{e} + \tilde{\mathbf{m}} \cdot \mathbf{e} \otimes e^{-2\alpha} - \mathbf{n} \cdot \mathbf{e} \otimes \xi e^{-2\alpha}, \\
\Delta\tilde{\mathbf{n}} \cdot \mathbf{e} &= 1 \otimes \tilde{\mathbf{n}} \cdot \mathbf{e} + \tilde{\mathbf{n}} \cdot \mathbf{e} \otimes e^{-2\alpha} + \mathbf{n} \cdot \mathbf{e} \otimes \eta e^{-2\alpha}
\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}
S(\alpha) &= -\alpha, \\
S(z) &= -e^{-2\alpha} z, \\
S(\mathbf{n} \cdot \mathbf{e}) &= -\mathbf{n} \cdot \mathbf{e}, \\
S(\tilde{\mathbf{m}} \cdot \mathbf{e}) &= -\tilde{\mathbf{m}} \cdot \mathbf{e} e^{2\alpha} + \mathbf{n} \cdot \mathbf{e} \xi, \\
S(\tilde{\mathbf{n}} \cdot \mathbf{e}) &= -\tilde{\mathbf{n}} \cdot \mathbf{e} e^{2\alpha} - \mathbf{n} \cdot \mathbf{e} \eta.
\end{aligned} \tag{5.25}$$

5.3 Classical basis

In potential applications of the bicrossproduct $\mathbb{C}(AN) \bowtie U(\mathfrak{h})$ in physics (particularly in the Lorentzian case with a timelike deformation parameter, where this bicrossproduct is the standard version of the κ -Poincaré algebra), it is convenient to work in a different basis, often called the classical basis [29]. This basis also turns out to be convenient for studying the associated Lie bi-algebra and we therefore write our results in terms of classical bases, adapted to the sign of the cosmological constant.

In order to match the usual conventions, we first have to replace α and z by their antipodes $\tilde{\alpha} = S(\alpha) = -\alpha$, $\tilde{z} = S(z) = -ze^{-2\alpha}$. The antipode S defined in (4.6) becomes

$$S(\mathbf{q}) = (\tilde{w}_3, \tilde{\xi}\mathbf{m} + \tilde{\eta}(\mathbf{m} \times \mathbf{n}) + \frac{1}{\lambda}(\tilde{w}_3 - e^{-2\tilde{\alpha}})\mathbf{n}), \tag{5.26}$$

where

$$\tilde{w}_3 = \cosh 2\alpha \pm \frac{1}{2}(\tilde{\xi}^2 - \lambda\tilde{\eta}^2)e^{-2\tilde{\alpha}} \tag{5.27}$$

and $\tilde{z} = \tilde{\xi} + \theta\tilde{\eta}$. The classical generators $\mathcal{P}_n, \mathcal{P}_{mn}$ and \mathcal{P}_m are then related to the bicrossproduct basis generators according to

$$\begin{aligned}\mathcal{P}_m &= \tilde{\xi}, & \mathcal{P}_{mn} &= \tilde{\eta}, & \mathcal{P}_3 &= \tilde{w}_3 = \lambda\mathcal{P}_n + e^{2\tilde{\alpha}}, \\ \mathcal{P}_n &= \frac{1}{\lambda} \left(\sinh 2\tilde{\alpha} \pm \frac{1}{2}(\tilde{\xi}^2 - \lambda\tilde{\eta}^2)e^{-2\tilde{\alpha}} \right),\end{aligned}\tag{5.28}$$

and satisfy the constraint

$$\mathcal{P}_3^2 \mp \mathcal{P}_m^2 \pm \lambda\mathcal{P}_{mn}^2 - \lambda^2\mathcal{P}_n^2 = 1.\tag{5.29}$$

From these, we obtain the algebra

$$\begin{aligned}[\mathcal{P}_{mn}, (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}] &= [\mathcal{P}_m, \lambda\mathbf{m} \cdot \mathbf{e}] = [\mathcal{P}_n, \mathbf{n} \cdot \mathbf{e}] = 0, \\ [\mathcal{P}_m, \mathbf{n} \cdot \mathbf{e}] &= -2\lambda\mathcal{P}_{mn}, & [\mathcal{P}_{mn}, \mathbf{n} \cdot \mathbf{e}] &= -2\mathcal{P}_m, \\ [\mathcal{P}_n, (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}] &= \mp 2\mathcal{P}_m, & [\mathcal{P}_m, (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}] &= 2\mathcal{P}_n, \\ [\mathcal{P}_n, \lambda\mathbf{m} \cdot \mathbf{e}] &= \pm 2\lambda\mathcal{P}_{mn}, & [\mathcal{P}_{mn}, \lambda\mathbf{m} \cdot \mathbf{e}] &= 2\mathcal{P}_n.\end{aligned}\tag{5.30}$$

together with the commutators (2.32).

Defining the vector

$$\mathcal{P} = \mathcal{P}_n \mathbf{n} + \mathcal{P}_m \mathbf{m} + \mathcal{P}_{mn} \mathbf{m} \times \mathbf{n},\tag{5.31}$$

we have

$$\mathbf{n} \cdot \mathcal{P} = -\lambda\mathcal{P}_n, \quad \mathbf{m} \cdot \mathcal{P} = \mp \frac{1}{\lambda}\mathcal{P}_m, \quad (\mathbf{m} \times \mathbf{n}) \cdot \mathcal{P} = \pm\mathcal{P}_{mn}.\tag{5.32}$$

In terms of the components \mathcal{P}_a , $a = 0, 1, 2$ of \mathcal{P} with respect to any orthonormal basis, and the generators $J_a = \frac{1}{2}e_a$, the brackets (5.30) become the ‘classical’ brackets

$$[J_a, J_b] = \epsilon_{abc}J^c, \quad [J_a, \mathcal{P}_b] = \epsilon_{abc}\mathcal{P}^c, \quad [\mathcal{P}_a, \mathcal{P}_b] = 0,\tag{5.33}$$

and the constraint (5.29) is simply

$$\mathcal{P}_3^2 + \lambda\mathcal{P}^2 = 1.\tag{5.34}$$

In order to write down the co-algebra, we introduce

$$\vec{\mathcal{P}}^2 = \mathcal{P}_m^2 - \lambda\mathcal{P}_{mn}^2, \quad \text{and} \quad T = e^{2\tilde{\alpha}} = \left[1 + \lambda^2\mathcal{P}_n^2 \pm \vec{\mathcal{P}}^2 \right]^{\frac{1}{2}} - \lambda\mathcal{P}_n.\tag{5.35}$$

Then the co-products of the classical generators are given by

$$\begin{aligned}\Delta\mathcal{P}_n &= \frac{1}{2\lambda} (T \otimes T - T^{-1} \otimes T^{-1}) \\ &\quad \pm \frac{1}{2\lambda} \left(T^{-1}\vec{\mathcal{P}}^2 \otimes T + 2T^{-1}(\mathcal{P}_m \otimes \mathcal{P}_m - \lambda\mathcal{P}_{mn} \otimes \mathcal{P}_{mn}) + T^{-1} \otimes T^{-1}\vec{\mathcal{P}}^2 \right), \\ \Delta\mathcal{P}_m &= \mathcal{P}_m \otimes T + 1 \otimes \mathcal{P}_m, \\ \Delta\mathcal{P}_{mn} &= \mathcal{P}_{mn} \otimes T + 1 \otimes \mathcal{P}_{mn}, \\ \Delta\mathbf{n} \cdot \mathbf{e} &= 1 \otimes \mathbf{n} \cdot \mathbf{e} + \mathbf{n} \cdot \mathbf{e} \otimes 1, \\ \Delta(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} &= 1 \otimes (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} + (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} \otimes T \pm \mathbf{n} \cdot \mathbf{e} \otimes \mathcal{P}_{mn}, \\ \Delta\lambda\mathbf{m} \cdot \mathbf{e} &= 1 \otimes \lambda\mathbf{m} \cdot \mathbf{e} + \lambda\mathbf{m} \cdot \mathbf{e} \otimes T \mp \mathbf{n} \cdot \mathbf{e} \otimes \mathcal{P}_m.\end{aligned}\tag{5.36}$$

When $\lambda = 0$, the classical basis is related to the bicrossproduct basis by

$$\begin{aligned}\mathcal{P}_n &= e^{-2\alpha} \left(\eta + \frac{1}{2} \xi^2 \right), \\ \mathcal{P}_{\tilde{m}} &= \xi \quad \mathcal{P}_{\tilde{n}} = e^{2\alpha} - 1.\end{aligned}\tag{5.37}$$

In analogy to (5.31), we define the vector

$$\mathcal{P} = \mathcal{P}_n \mathbf{n} + \mathcal{P}_{\tilde{m}} \tilde{\mathbf{m}} + \mathcal{P}_{\tilde{n}} \tilde{\mathbf{n}},\tag{5.38}$$

so that

$$\mathbf{n} \cdot \mathcal{P} = \mathcal{P}_{\tilde{n}}, \quad \tilde{\mathbf{m}} \cdot \mathcal{P} = -\mathcal{P}_{\tilde{m}}, \quad \tilde{\mathbf{n}} \cdot \mathcal{P} = \mathcal{P}_n.\tag{5.39}$$

In this basis, we obtain the Lie algebra

$$\begin{aligned}[\mathcal{P}_{\tilde{n}}, \mathbf{n} \cdot \mathbf{e}] &= [\mathcal{P}_{\tilde{m}}, \tilde{\mathbf{m}} \cdot \mathbf{e}] = [\mathcal{P}_n, \tilde{\mathbf{n}} \cdot \mathbf{e}] = 0, \\ [\mathcal{P}_n, \mathbf{n} \cdot \mathbf{e}] &= -2\mathcal{P}_{\tilde{m}}, \quad [\mathcal{P}_{\tilde{m}}, \mathbf{n} \cdot \mathbf{e}] = -2\mathcal{P}_{\tilde{n}}, \\ [\mathcal{P}_{\tilde{n}}, \tilde{\mathbf{n}} \cdot \mathbf{e}] &= 2\mathcal{P}_{\tilde{m}}, \quad [\mathcal{P}_n, \tilde{\mathbf{m}} \cdot \mathbf{e}] = 2\mathcal{P}_n, \\ [\mathcal{P}_{\tilde{n}}, \tilde{\mathbf{m}} \cdot \mathbf{e}] &= -2\mathcal{P}_{\tilde{n}}, \quad [\mathcal{P}_{\tilde{m}}, \tilde{\mathbf{n}} \cdot \mathbf{e}] = 2\mathcal{P}_n,\end{aligned}\tag{5.40}$$

together with (2.37), which is again equivalent to (5.33).

With $\tilde{T} = e^{2\alpha} = 1 + \mathcal{P}_{\tilde{n}} = 1 + \mathbf{n} \cdot \mathcal{P}$, the co-products are

$$\begin{aligned}\Delta \mathcal{P}_n &= 1 \otimes \mathcal{P}_n + \mathcal{P}_n \otimes \tilde{T}^{-1} + \frac{1}{2} \mathcal{P}_{\tilde{n}} \otimes \tilde{T}^{-1} \mathcal{P}_{\tilde{m}}^2 + \mathcal{P}_{\tilde{m}} \otimes \tilde{T}^{-1} \mathcal{P}_{\tilde{m}}, \\ \Delta \mathcal{P}_{\tilde{m}} &= \mathcal{P}_{\tilde{m}} \otimes 1 + \tilde{T} \otimes \mathcal{P}_{\tilde{m}}, \\ \Delta \mathcal{P}_{\tilde{n}} &= \mathcal{P}_{\tilde{n}} \otimes \mathcal{P}_{\tilde{n}}, \\ \Delta \mathbf{n} \cdot \mathbf{e} &= 1 \otimes \mathbf{n} \cdot \mathbf{e} + \mathbf{n} \cdot \mathbf{e} \otimes 1, \\ \Delta \tilde{\mathbf{m}} \cdot \mathbf{e} &= 1 \otimes \tilde{\mathbf{m}} \cdot \mathbf{e} + \tilde{\mathbf{m}} \cdot \mathbf{e} \otimes \tilde{T}^{-1} - \mathbf{n} \cdot \mathbf{e} \otimes \tilde{T}^{-1} \mathcal{P}_{\tilde{m}}, \\ \Delta \tilde{\mathbf{n}} \cdot \mathbf{e} &= 1 \otimes \tilde{\mathbf{n}} \cdot \mathbf{e} + \tilde{\mathbf{n}} \cdot \mathbf{e} \otimes \tilde{T}^{-1} + \mathbf{n} \cdot \mathbf{e} \otimes \mathcal{P}_n - \frac{1}{2} \mathbf{n} \cdot \mathbf{e} \otimes \tilde{T}^{-1} \mathcal{P}_{\tilde{m}}^2.\end{aligned}\tag{5.41}$$

5.4 Summary of results

In Table 2 we list the semiduals computed in this paper of the universal enveloping algebras of the Lie algebras for the groups and factorisation given in Table 1. In the Euclidean case with $\lambda > 0$, semidualisation of the enveloping algebra $U(\mathfrak{su}(2) \bowtie \mathfrak{su}(2))$ gives the quantum double $D(U(\mathfrak{su}(2))) = \mathbb{C}(SU(2)) \bowtie U(\mathfrak{su}(2))$. For $\lambda = 0$ and Euclidean signature, the semidual of $U(\mathfrak{su}(2) \bowtie \mathbb{R}^3)$ is $(\mathbb{R}^*)^3 \bowtie U(\mathfrak{su}(2))$. In the Lorentzian case and with $\lambda > 0$, semidualisation of $U(\mathfrak{sl}(2, \mathbb{R}) \bowtie U(\mathfrak{sl}(2, \mathbb{R})))$ gives the quantum double $D(U(\mathfrak{sl}(2, \mathbb{R}))) = \mathbb{C}(SL(2, \mathbb{R})) \bowtie U(\mathfrak{sl}(2, \mathbb{R}))$. When $\lambda = 0$, the semidual of $U(\mathfrak{sl}(2, \mathbb{R}) \bowtie \mathbb{R}^3)$ is $(\mathbb{R}^*)^3 \bowtie U(\mathfrak{sl}(2, \mathbb{R}))$. All this follows from our discussion of examples in Sect. 3.2. We do not give details of the (standard) quantum group structure for these cases, but see [13] for details of the Euclidean case. For the (non-trivial) bicrossproducts in Table 2 we use the notation \blacktriangleright_s , \blacktriangleright_l and \blacktriangleright_t for the left-right bicrossproducts with spacelike, lightlike and timelike deformations respectively. In order to make contact with the standard literature on κ -Poincaré algebras and related bicrossproducts we discuss each of these in conventional notation in the Appendix. The first treatments of κ -Poincaré symmetry focused on the Lorentzian case, with a timelike deformation vector [15, 16] but spacelike [30] and lightlike [31] deformation vectors were soon considered. In the Euclidean case different choices of \mathbf{n} lead to isomorphic Lie bialgebras, see e.g. Chapter 8 of [18] for a details.

| | Euclidean signature | Lorentzian signature |
|---------------|---|---|
| $\lambda > 0$ | $D(U(\mathfrak{su}(2)))$ | $D(U(\mathfrak{sl}(2, \mathbb{R})))$ $\mathbb{C}(AN(2)) \blacktriangleright_s U(\mathfrak{sl}(2, \mathbb{R}))$ |
| $\lambda = 0$ | $(\mathbb{R}^*)^3 \rtimes U(\mathfrak{su}(2))$ | $(\mathbb{R}^*)^3 \rtimes U(\mathfrak{sl}(2, \mathbb{R}))$ $\mathbb{C}(AN(2)) \blacktriangleright_l U(\mathfrak{sl}(2, \mathbb{R}))$ |
| $\lambda < 0$ | $\mathbb{C}(AN(2)) \blacktriangleright U(\mathfrak{su}(2))$ | $\mathbb{C}(AN(2)) \blacktriangleright_t U(\mathfrak{sl}(2, \mathbb{R}))$ |

Table 2: Semiduals of local isometry groups in 3d gravity

6 Discussion and conclusion

6.1 Classical r -matrices

Before we discuss our results we consider the Lie bi-algebra structures for each of the bi-crossproduct Hopf algebras we have obtained. We work in the classical basis of Sect. 5.3 where the Lie brackets have the form (5.33). This shows in particular that, for all values of the cosmological constant, the Lie algebra is $\mathfrak{g} = \mathbb{R}^3 \rtimes \mathfrak{h}$, which is the Lie algebra of the Euclidean group in the Euclidean case, and the Lie algebra of the Poincaré group in the Lorentzian case.

In order to extract the co-commutator for a generator Y , we compute $\Delta(Y) - \Delta^{op}(Y)$ for each of the co-products in (5.36) for $\lambda \neq 0$ and in (5.41) for $\lambda = 0$, and keep only leading (quadratic) terms. For $\lambda \neq 0$, we find

$$\begin{aligned}
\delta(\mathbf{n} \cdot \mathcal{P}) &= \delta(\mathbf{n} \cdot \mathbf{e}) = 0, \\
\delta(\mathbf{m} \cdot \mathcal{P}) &= \mathbf{n} \cdot \mathcal{P} \wedge \mathbf{m} \cdot \mathcal{P}, \\
\delta((\mathbf{m} \times \mathbf{n}) \cdot \mathcal{P}) &= \mathbf{n} \cdot \mathcal{P} \wedge (\mathbf{m} \times \mathbf{n}) \cdot \mathcal{P}, \\
\delta(\mathbf{m} \cdot \mathbf{e}) &= \mathbf{n} \cdot \mathcal{P} \wedge \mathbf{m} \cdot \mathbf{e} + \mathbf{n} \cdot \mathbf{e} \wedge \mathbf{m} \cdot \mathcal{P}, \\
\delta((\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}) &= \mathbf{n} \cdot \mathcal{P} \wedge (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e} + \mathbf{n} \cdot \mathbf{e} \wedge (\mathbf{m} \times \mathbf{n}) \cdot \mathcal{P},
\end{aligned} \tag{6.1}$$

with \wedge denoting the skewsymmetric tensor product. For $\lambda = 0$, we obtain

$$\begin{aligned}
\delta(\mathbf{n} \cdot \mathcal{P}) &= \delta(\mathbf{n} \cdot \mathbf{e}) = 0, \\
\delta(\tilde{\mathbf{n}} \cdot \mathcal{P}) &= \mathbf{n} \cdot \mathcal{P} \wedge \tilde{\mathbf{n}} \cdot \mathcal{P}, \\
\delta(\tilde{\mathbf{m}} \cdot \mathcal{P}) &= \mathbf{n} \cdot \mathcal{P} \wedge \tilde{\mathbf{m}} \cdot \mathcal{P}, \\
\delta(\tilde{\mathbf{n}} \cdot \mathbf{e}) &= \mathbf{n} \cdot \mathcal{P} \wedge \tilde{\mathbf{n}} \cdot \mathbf{e} + \mathbf{n} \cdot \mathbf{e} \wedge \tilde{\mathbf{n}} \cdot \mathcal{P}, \\
\delta(\tilde{\mathbf{m}} \cdot \mathbf{e}) &= \mathbf{n} \cdot \mathcal{P} \wedge \tilde{\mathbf{m}} \cdot \mathbf{e} + \mathbf{n} \cdot \mathbf{e} \wedge \tilde{\mathbf{m}} \cdot \mathcal{P}.
\end{aligned} \tag{6.2}$$

A co-commutator for a Lie algebra \mathfrak{g} is co-boundary if its action on any $Y \in \mathfrak{g}$ can be written as

$$\delta(Y) = [1 \otimes Y + Y \otimes 1, r], \tag{6.3}$$

in terms of an element $r \in \mathfrak{g} \otimes \mathfrak{g}$, called the classical r -matrix. Both the co-commutators (6.1) and (6.2) turn out to be co-boundary. The r -matrix in both cases is [32]

$$r_{\mathbf{n}} = -\epsilon_{abc} n^c J^a \wedge \mathcal{P}^b. \quad (6.4)$$

This is the classical r -matrix of the 3d κ -Poincaré (or Euclidean) algebra with deformation vector \mathbf{n} .

For completeness we also note the classical r -matrices for the quantum doubles $D(U(\mathfrak{su}(2)))$ and $D(U(\mathfrak{sl}(2, \mathbb{R})))$ in Table (2), which arise when $\lambda > 0$. These are well-known [9, 13] and easily computed. In the semiclassical basis used in [13] one deduces the co-product from the multiplication rule of quaternions of the form $P_0 + \sqrt{\lambda} \mathbf{e} \cdot \mathcal{P}$. Extracting leading terms one finds the co-commutator

$$\delta(\mathcal{P}_a) = \sqrt{\lambda} \epsilon_{abc} \mathcal{P}^b \wedge \mathcal{P}^c, \quad (6.5)$$

valid in both the Euclidean and Lorentzian setting. This is co-boundary, with classical r -matrix

$$r_D = \sqrt{\lambda} \mathcal{P}_a \wedge J^a, \quad (6.6)$$

where we have omitted symmetric, invariant terms.

6.2 Interpretation in the context of 3d gravity

We have seen that the algebra structure of the Hopf algebra we obtain via semidualisation is independent of the cosmological parameter λ and always that of a universal enveloping algebra: that of the Euclidean Lie algebra in the Euclidean case, and that of the Poincaré Lie algebra in the Lorentzian case. However, the co-algebra structure depends both on the original isometry group (i.e. on λ) and on the factorisation (i.e. on the vector \mathbf{n} in the cases where it is defined).

In interpreting our results it is important to keep in mind the distinction between the original isometry groups with the Lie algebras \mathfrak{g}_λ generated by J_a and P_a , $a = 0, 1, 2$, and the semidual Lie brackets (5.33) of the Euclidean or Poincaré Lie algebra with generators J_a and \mathcal{P}_a , $a = 0, 1, 2$. The ‘dual momenta’ \mathcal{P}_a generate the (abelian) algebra dual to that of the original momentum algebra $U(\mathfrak{an}(2)_{\mathbf{n}})$ with generators \tilde{P}_a , $a = 0, 1, 2$ (2.10). From the point of view of the original symmetry algebras one should really think of the \mathcal{P}_a as coordinates on position space: the constraint (5.34) is precisely the constraint (2.39) defining the model spacetimes. However, in keeping with the philosophy and terminology of [13] we combine the mathematical operation of semidualisation with a shift of interpretation: we think of the \mathcal{P}_a as momentum generators in a dual theory, and ask how the applicability regimes of the original and dual theory are related. In this way, our interpretation of the generators \mathcal{P}_a also agrees with that in the physics literature on κ -Poincaré or quantum double symmetry.

In order to characterise and distinguish the original and the semidual theory, it is worth recalling [33] that in 3d gravity the curvature of the model spacetime is controlled by the cosmological constant, with ‘curvature radius’

$$\tau_C = \frac{1}{\sqrt{\lambda}}, \quad (6.7)$$

where we used the letter τ_C to remind the reader that τ_C has the physical dimension of time, since the cosmological constant Λ_C has physical dimension of inverse length squared and $\lambda = -c^2\Lambda_C$. The curvature of momentum space, on the other hand, is controlled by Newton's constant G , with the curvature radius given by the 3d Planck mass:

$$\mu_p = \frac{1}{16\pi G}. \quad (6.8)$$

Note that our definition of the Planck mass differs by a factor of two from the one which is most commonly used in 3d gravity; we adopt it here merely to simplify expressions in the following general discussion

The above summary is a useful guideline even though it is strictly speaking only true when either the cosmological constant or Newton's constant is zero; in general the two parameters combine into a new dimensionless parameter which affects both spacetime and momentum curvature [5]. As further discussed in [13] and with the same caveat, the cosmological constant controls the algebra structure, while the Planck mass controls the co-algebra structure in the quantum isometry groups of 3d quantum gravity. One should thus think of the universal enveloping algebras $U(\mathfrak{g}_\lambda)$ as quantum isometry groups in a regime with flat momentum space (and hence infinite Planck mass) and curved spacetime, controlled by the cosmological parameter λ . By contrast, the semidual quantum groups are associated to flat model spacetimes (since the 'semidual cosmological constant' is zero in the brackets (5.33)) but curved momentum space (since the co-algebra structures are non-trivial).

We now want to relate the cosmological parameter λ of the original theory to the Planck mass $\tilde{\mu}_P$ in the semidual theory. To do this, we briefly recall aspects of the combinatorial quantisation programme which we touched on in the introduction. This programme is based on the Chern-Simons formulation of 3d gravity, where the local isometry groups play the role of gauge groups. The Chern-Simons action requires an invariant, non-degenerate inner product on the Lie algebra of the relevant local isometry group. In order to write the Einstein-Hilbert action as a Chern-Simons action (up to boundary terms) for the (semidual) Lie algebra spanned by J_a, \mathcal{P}_a , $a = 1, 2, 3$, one needs to take the inner product with non-vanishing pairings

$$\langle J_a, \mathcal{P}_b \rangle = \frac{1}{16\pi\tilde{G}} \eta_{ab}, \quad (6.9)$$

where \tilde{G} is Newton's constant in the semidual theory, again related to the Planck mass via $\tilde{\mu}_p = 1/(16\pi\tilde{G})$.

The combinatorial quantisation programme is based on a description of the Poisson structure on an extended phase space, due to Fock and Rosly [34]. The latter makes essential use of a classical r -matrix which is required to be compatible with the inner product (6.9) in the sense that it satisfies the classical Yang-Baxter equation and its symmetric part is equal to the Casimir

$$K = \frac{1}{\tilde{\mu}_P} (J_a \otimes \mathcal{P}^a + \mathcal{P}_a \otimes J^a) \quad (6.10)$$

associated to (6.9).

All the r -matrices we computed in Sect. 6.1 are antisymmetric. To ascertain their compatibility with 3d gravity we therefore need to check if their sum with the Casimir (6.10)

satisfies the classical Yang-Baxter equation. This was systematically investigated in [32]. The result is that the r -matrix $r_{\mathbf{n}}$ (6.4) of bicrossproduct type is compatible with the inner product (6.9) iff

$$\mathbf{n}^2 = -\frac{1}{\tilde{\mu}_P^2}. \quad (6.11)$$

Using $\mathbf{n}^2 = -\lambda$ we deduce the condition

$$\lambda = \frac{1}{\tilde{\mu}_P^2}. \quad (6.12)$$

Similarly, the r -matrix (6.6) of the doubles $D(U(\mathfrak{su}(2)))$ and $D(U(\mathfrak{sl}(2, \mathbb{R})))$ is compatible with the Casimir (6.10) if $\sqrt{\lambda} = 1/\tilde{\mu}_P$. Since $\lambda > 0$ by assumption in this case, this requirement is equivalent to the condition (6.12), which therefore covers all cases.

The schematic summary of our results in Table 3 shows that semidualisation may be viewed as an exchange $\mu_P \leftrightarrow \tau_C$, confirming and extending the result obtained in the Euclidean setting in [13]. The mismatch of physical dimensions in this exchange is a consequence of the exchange of position and momentum degrees of freedom under the semidualisation map. In the models considered here (where either the cosmological time scale or the Planck mass is infinite) this has to be repaired ‘by hand’, by multiplying with a suitable dimensionful constant. To understand this better one needs to go to a regime where *both* the cosmological time scale and the Planck mass have finite values. The theory in that regime is controlled by a dimensionless parameter, essentially an exponential of $\hbar/(\mu_P \tau_C)$, which is manifestly invariant under the exchange of cosmological time scale and Planck mass [13]. The limits $\mu_P \rightarrow \infty$ and $\tau_C \rightarrow \infty$ correspond to different contractions of the associated quantum isometry groups. The contractions require the introduction of dimensionful constants, and this is the origin of the different physical dimensions of the remaining parameter after contraction.

| | Original regime | Semidual regime |
|-------------------------|-------------------------------------|---------------------------|
| Cosmological time scale | $\tau_C = \frac{1}{\sqrt{\lambda}}$ | $\tilde{\tau}_C = \infty$ |
| Planck mass | $\mu_P = \infty$ | $\tilde{\mu}_P = \tau_C$ |

Table 3: Semiduality of regimes in 3d gravity.

While the parameter λ naturally takes any real value, a physically reasonable Planck mass should be real. At first sight it is therefore a concern that the condition (6.12) may force the semidual Planck mass to be imaginary. However, formally extending our interpretation of 3d gravity to include imaginary Planck mass we obtain a very symmetric picture, and a gravitational interpretation of all quantum groups constructed in this paper via semiduality. In particular, one could then rephrase the result of [32] that the κ -Poincaré algebra with a

timelike deformation vector \mathbf{n} is not associated to 3d quantum gravity by saying that the κ -Poincaré algebra with a timelike deformation vector \mathbf{n} describes 3d quantum gravity with an imaginary Planck mass.

Finally, we note that pairs of bicrossproducts which arise, via semiduality, from two different factorisations of the same isometry group have twist-equivalent associated Lie bialgebras. This follows from the fact that they are equivalent as algebras and that their associated classical r -matrices are compatible with the same inner product (6.9). The most interesting example of such a pair of bicrossproducts is the quantum double $D(U(\mathfrak{sl}(2, \mathbb{R})))$ and the bicrossproduct $\mathbb{C}(AN(2)) \blacktriangleright_s U(\mathfrak{sl}(2, \mathbb{R}))$ with a spacelike deformation parameter, which are therefore equally valid quantum isometry groups in the combinatorial factorisation programme. Here the consideration of their semiduals provides a link between two quantum groups whose relation would otherwise be somewhat mysterious.

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A The results in conventional notations

The quaternionic formalism used in this paper is very effective for our purposes but leads to a description of bicrossproduct Hopf algebras like the κ -Poincaré and κ -Euclidean algebras which looks rather different from that given in the standard literature. In this appendix we therefore summarise the algebra and co-algebra structure of each of the bicrossproduct Hopf algebras in Table 2 in conventional notation.

A.1 The case $\lambda \neq 0$

For the case $\lambda < 0$ and $c^2 < 0$ (Euclidean signature), the semidual Hopf algebra in Table 2 is the Euclidean bicrossproduct $\mathbb{C}(AN(2)) \blacktriangleright U(\mathfrak{su}(2))$. The translation generators $p_a, a = 0, 1, 2$, rotation M and boosts $N_i, i = 1, 2$, are given by

$$\begin{aligned} 2\alpha &= \frac{p_0}{\sqrt{-\lambda}}, & \xi &= \frac{p_2}{\sqrt{-\lambda}}, & \eta &= \frac{p_1}{\lambda}, \\ \text{and } M &= \frac{1}{2\sqrt{-\lambda}} \mathbf{n} \cdot \mathbf{e}, & N_1 &= -\frac{1}{2} (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}, & N_2 &= \frac{\sqrt{-\lambda}}{2} \mathbf{m} \cdot \mathbf{e}. \end{aligned} \quad (\text{A.1})$$

This gives the algebra

$$\begin{aligned} [p_a, p_b] &= 0, & [M, N_1] &= -N_2, & [M, N_2] &= N_1, & [N_1, N_2] &= -M, \\ [p_0, M] &= 0 & [p_i, M] &= \epsilon_{ij} p_j, \\ [p_0, N_i] &= -\epsilon_{ij} p_j e^{-\frac{p_0}{\sqrt{-\lambda}}}, \\ [p_i, N_j] &= -\epsilon_{ij} e^{-\frac{p_0}{\sqrt{-\lambda}}} \left(\frac{\sqrt{-\lambda}}{2} (e^{\frac{2p_0}{\sqrt{-\lambda}}} - 1) - \frac{1}{2\sqrt{-\lambda}} p^2 \right), & i, j &= 1, 2, \end{aligned} \quad (\text{A.2})$$

where $p^2 = p_1^2 + p_2^2$. The co-products are

$$\begin{aligned}
\Delta p_0 &= p_0 \otimes 1 + 1 \otimes p_0, \\
\Delta p_i &= p_i \otimes 1 + e^{\frac{p_0}{\sqrt{-\lambda}}} \otimes p_i, \\
\Delta M &= 1 \otimes M + M \otimes 1, \\
\Delta N_i &= 1 \otimes N_i + N_i \otimes e^{-\frac{p_0}{\sqrt{-\lambda}}} + \frac{1}{\sqrt{-\lambda}} M \otimes p_i e^{-\frac{p_0}{\sqrt{-\lambda}}}, \quad i = 1, 2.
\end{aligned} \tag{A.3}$$

One can match the conventions of the algebra described in [14] by choosing basis in which

$$\tilde{p}_i = p_i e^{-\frac{p_0}{\sqrt{-\lambda}}}. \tag{A.4}$$

In this case, we obtain the algebra

$$\begin{aligned}
[\tilde{p}_a, \tilde{p}_b] &= 0, \quad [M, N_1] = -N_2, \quad [M, N_2] = N_1, \quad [N_1, N_2] = -M, \\
[\tilde{p}_0, M] &= 0 \quad [\tilde{p}_i, M] = \epsilon_{ij} \tilde{p}_j, \\
[p_0, N_i] &= -\epsilon_{ij} \tilde{p}_j, \quad i, j = 1, 2, \\
[\tilde{p}_1, N_1] &= \frac{1}{\sqrt{-\lambda}} \tilde{p}_1 \tilde{p}_2, \quad [\tilde{p}_2, N_2] = -\frac{1}{\sqrt{-\lambda}} \tilde{p}_1 \tilde{p}_2, \\
[\tilde{p}_1, N_2] &= -\frac{\sqrt{-\lambda}}{2} (1 - e^{-\frac{2p_0}{\sqrt{-\lambda}}}) + \frac{1}{2\sqrt{-\lambda}} (\tilde{p}_2^2 - \tilde{p}_1^2), \\
[\tilde{p}_2, N_1] &= \frac{\sqrt{-\lambda}}{2} (1 - e^{-\frac{2p_0}{\sqrt{-\lambda}}}) + \frac{1}{2\sqrt{-\lambda}} (\tilde{p}_2^2 - \tilde{p}_1^2).
\end{aligned} \tag{A.5}$$

with co-products

$$\begin{aligned}
\Delta p_0 &= p_0 \otimes 1 + 1 \otimes p_0, \\
\Delta \tilde{p}_i &= \tilde{p}_i \otimes e^{-\frac{p_0}{\sqrt{-\lambda}}} + 1 \otimes \tilde{p}_i, \quad i = 1, 2 \\
\Delta M &= 1 \otimes M + M \otimes 1, \\
\Delta N_i &= 1 \otimes N_i + N_i \otimes e^{-\frac{p_0}{\sqrt{-\lambda}}} + \frac{1}{\sqrt{-\lambda}} M \otimes \tilde{p}_i, \quad i = 1, 2.
\end{aligned} \tag{A.6}$$

When $\lambda < 0$ and $c^2 > 0$ (Lorentzian signature), the bicrossproduct in Table 2 $\mathbb{C}(AN(2)) \blacktriangleright_{\mathbf{t}} U(\mathfrak{sl}(2, \mathbb{R}))$ (κ -Poincaré algebra with a timelike deformation parameter). Setting

$$\begin{aligned}
2\alpha &= \frac{p_0}{\sqrt{-\lambda}}, \quad \xi = \frac{p_2}{\sqrt{-\lambda}}, \quad \eta = \frac{p_1}{\lambda}, \\
\text{and } M &= \frac{1}{2\sqrt{-\lambda}} \mathbf{n} \cdot \mathbf{e}, \quad N_1 = -\frac{1}{2} (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}, \quad N_2 = \frac{\sqrt{-\lambda}}{2} \mathbf{m} \cdot \mathbf{e},
\end{aligned} \tag{A.7}$$

we have the algebra

$$\begin{aligned}
[p_a, p_b] &= 0, \quad [M, N_1] = -N_2, \quad [M, N_2] = N_1, \quad [N_1, N_2] = -M, \\
[p_0, M] &= 0 \quad [p_i, M] = \epsilon_{ij} p_j, \quad i, j = 1, 2 \\
[p_0, N_i] &= -\epsilon_{ij} p_j e^{-\frac{p_0}{\sqrt{-\lambda}}}, \quad i = 1, 2, \\
[p_i, N_j] &= -\epsilon_{ij} e^{-\frac{p_0}{\sqrt{-\lambda}}} \left(\frac{\sqrt{-\lambda}}{2} (e^{\frac{2p_0}{\sqrt{-\lambda}}} - 1) + \frac{1}{2\sqrt{-\lambda}} p^2 \right), \quad i = 1, 2,
\end{aligned} \tag{A.8}$$

where $p^2 = p_1^2 + p_2^2$. The co-products are

$$\begin{aligned}
\Delta p_0 &= p_0 \otimes 1 + 1 \otimes p_0, \\
\Delta p_i &= p_i \otimes 1 + e^{\frac{p_0}{\sqrt{-\lambda}}} \otimes p_i, \\
\Delta M &= 1 \otimes M + M \otimes 1, \\
\Delta N_i &= 1 \otimes N_i + N_i \otimes e^{-\frac{p_0}{\sqrt{-\lambda}}} - \frac{1}{\sqrt{-\lambda}} M \otimes p_i e^{-\frac{p_0}{\sqrt{-\lambda}}}, \quad i = 1, 2.
\end{aligned} \tag{A.9}$$

For the case $\lambda > 0$ and $c^2 > 0$ (Lorentzian signature), the bicrossproduct in Table 2 is $\mathbb{C}(AN(2)) \blacktriangleright_{\mathfrak{s}} U(\mathfrak{sl}(2, \mathbb{R}))$ (κ -Poincaré algebra with a spacelike deformation parameter). The translation generators $p_a, a = 0, 1, 2$, rotation M and boosts $N_i, i = 1, 2$, are given by

$$\begin{aligned}
2\alpha &= \frac{p_1}{\sqrt{\lambda}}, \quad \xi = \frac{p_2}{\sqrt{\lambda}}, \quad \eta = \frac{p_0}{\lambda}, \\
\text{and } N_1 &= \frac{1}{2\sqrt{\lambda}} \mathbf{n} \cdot \mathbf{e}, \quad M = -\frac{1}{2}(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}, \quad N_2 = \frac{\sqrt{\lambda}}{2} \mathbf{m} \cdot \mathbf{e},
\end{aligned} \tag{A.10}$$

from which (5.1) gives

$$\begin{aligned}
[p_a, p_b] &= 0, \quad [M, N_1] = -N_2, \quad [M, N_2] = N_1, \quad [N_1, N_2] = M, \\
[p_0, M] &= 0, \quad [p_1, M] = -p_2 e^{-\frac{p_1}{\sqrt{\lambda}}}, \quad [p_2, M] = e^{-\frac{p_1}{\sqrt{\lambda}}} \left(\frac{\sqrt{\lambda}}{2} (e^{\frac{2p_1}{\sqrt{\lambda}}} - 1) - \frac{1}{2\sqrt{\lambda}} (p_2^2 - p_0^2) \right) \\
[p_0, N_1] &= -p_2, \quad [p_0, N_2] = e^{-\frac{p_1}{\sqrt{\lambda}}} \left(\frac{\sqrt{\lambda}}{2} (e^{\frac{2p_1}{\sqrt{\lambda}}} - 1) - \frac{1}{2\sqrt{\lambda}} (p_2^2 - p_0^2) \right), \\
[p_1, N_1] &= [p_2, N_2] = 0, \quad [p_1, N_2] = p_0 e^{-\frac{p_1}{\sqrt{\lambda}}}, \quad [p_2, N_1] = -p_0.
\end{aligned} \tag{A.11}$$

The co-products are

$$\begin{aligned}
\Delta p_0 &= p_0 \otimes 1 + e^{\frac{p_1}{\sqrt{\lambda}}} \otimes p_0, \\
\Delta p_1 &= p_1 \otimes 1 + 1 \otimes p_1, \\
\Delta p_2 &= p_2 \otimes 1 + e^{\frac{p_1}{\sqrt{\lambda}}} \otimes p_2, \\
\Delta M &= 1 \otimes M + M \otimes e^{-\frac{p_1}{\sqrt{\lambda}}} - \frac{1}{\sqrt{\lambda}} N_1 \otimes p_0 e^{-\frac{p_1}{\sqrt{\lambda}}}, \\
\Delta N_1 &= 1 \otimes N_1 + N_1 \otimes 1, \\
\Delta N_2 &= 1 \otimes N_2 + N_2 \otimes e^{-\frac{p_1}{\sqrt{\lambda}}} - \frac{1}{\sqrt{\lambda}} N_1 \otimes p_2 e^{-\frac{p_1}{\sqrt{\lambda}}}.
\end{aligned} \tag{A.12}$$

A.2 The case $\lambda = 0$

For the case $\lambda = 0$ and $c^2 > 0$ (Lorentzian signature), the bicrossproduct in Table 2 is $\mathbb{C}(AN(2)) \blacktriangleright_l U(\mathfrak{sl}(2, \mathbb{R}))$ (κ -Poincaré algebra with lightlike deformation parameter). The translation generators p_-, p_+, \tilde{p} null rotations N_-, N_+ and boost \tilde{M} are

$$\begin{aligned}
2\alpha &= p_+, \quad \xi = \tilde{p}, \quad \eta = p_-, \\
\text{and } \tilde{M} &= \frac{1}{2} \tilde{\mathbf{m}} \cdot \mathbf{e}, \quad N_- = \frac{1}{2} \mathbf{n} \cdot \mathbf{e}, \quad N_+ = \frac{1}{2} \tilde{\mathbf{n}} \cdot \mathbf{e}.
\end{aligned} \tag{A.13}$$

This gives the algebra

$$\begin{aligned}
[\tilde{p}, p_-] &= [\tilde{p}, p_+] = [p_-, p_+] = 0, \\
[\tilde{M}, N_-] &= N_-, \quad [\tilde{M}, N_+] = -N_+, \quad [N_-, N_+] = -\tilde{M}, \\
[\tilde{p}, \tilde{M}] &= 0 \quad [p_-, \tilde{M}] = (p_- + \tfrac{1}{2}\tilde{p}^2)e^{-p_+}, \quad [p_+, \tilde{M}] = e^{-p_+} - 1, \\
[\tilde{p}, N_-] &= 1 - e^{p_+}, \quad [\tilde{p}, N_+] = (p_- + \tfrac{1}{2}\tilde{p}^2)e^{-p_+}, \\
[p_-, N_-] &= -\tilde{p}, \quad [p_-, N_+] = 0, \quad [p_+, N_-] = 0, \quad [p_+, N_+] = \tilde{p}e^{-p_+}.
\end{aligned} \tag{A.14}$$

The co-products are

$$\begin{aligned}
\Delta\tilde{p} &= \tilde{p} \otimes 1 + e^{p_+} \otimes \tilde{p}, \\
\Delta p_- &= p_- \otimes 1 + e^{p_+} \otimes p_-, \\
\Delta p_+ &= p_+ \otimes 1 + 1 \otimes p_+, \\
\Delta\tilde{M} &= 1 \otimes \tilde{M} - \tilde{M} \otimes e^{-p_+} - N_- \otimes \tilde{p}e^{-p_+}, \\
\Delta N_- &= 1 \otimes N_- + N_- \otimes 1, \\
\Delta N_+ &= 1 \otimes N_+ + N_+ \otimes e^{-p_+} + N_- \otimes p_- e^{-p_+}.
\end{aligned} \tag{A.15}$$

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